A SIMPLE PERTURBATION ALGORITHM FOR INVERTING THE CARTESIAN TO GEODETIC TRANSFORMATION

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A natural geometric perturbation variable is identified as the ratio of the major and minor Earth ellipse radii minus one. A singularity-free perturbation solution is presented for inverting the Cartesian to Geodetic transformation, which yields millimeter accuracy throughout the LEO through GEO range of satellite applications. Geocentric latitude is used to model the satellite ground track position vector. Rapidly converging perturbation solutions are developed for the satellite height above the Earth and the geocentric latitude as a perturbation power series in the geometric perturbation variable. Very compact series coefficients are recovered for the fourth order series approximations. The perturbation solution algorithm presented in this work provide three significant benefits over existing approaches for the problem: (1) No highly sensitive quartic polynomial solution algorithms are required; (2) A non-iterative algorithm inverts the transformation without requiring special starting guesses for the power series solution; and (3) Uniform solution accuracy is obtained for the Equator and the Polar regions. Simulation results are presented that compare the solution accuracy and algorithm performance for applications spanning the LEO-to-GEO range of missions.

INTRODUCTION

A frequent calculation for satellites in low Earth orbit (LEO) to geosynchronous Earth orbit (GEO) involves inverting transformations between 3D satellite Cartesian Earth centered coordinates and geodetic coordinates. The geodetic coordinates consist of λ_g, ϕ_g, and h, which denote the geodetic longitude of the satellite subpoint g, the geodetic latitude of the satellite, and the height of the satellite above the reference Earth elliptical surface along the surface normal from the geodetic ellipsoid to the satellite position. Referring to Figure 1, the transformation from geodetic $\left(\lambda_g, \phi_g, h\right)$ to Cartesian coordinates $\left(x_s, y_s, z_s\right)$ is given by

$$\begin{aligned} x_s &= (\mathrm{N}(\phi_g) + \mathrm{h})\cos\phi_g \cos\lambda_g \\ y_s &= (\mathrm{N}(\phi_g) + \mathrm{h})\cos\phi_g \sin\lambda_g \\ z_s &= (\mathrm{N}(\phi_g)(1 - \mathrm{e}^2) + \mathrm{h})\sin\phi_g \end{aligned} \tag{1}$$

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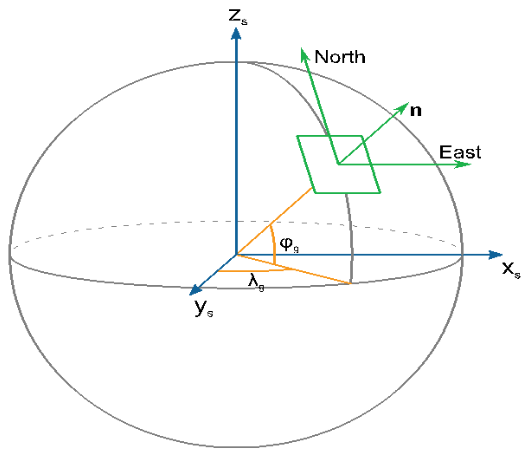

Figure 1: Geodetic and Cartesian Coordinates

where $N(\phi_g) = a/\sqrt{1-e^2\sin^2\phi_g}$ denotes the ellipsoid radius of curvature in the prime vertical plane defined by the vectors $\hat{n}$ (ellipsoid outward normal) and $\hat{\tau}$ (local east), h is assumed to lie along $\hat{n}$, a denotes the semi-major axis, b denotes the semi-minor axis, and e denotes the eccentricity of the Earth's reference ellipsoid.

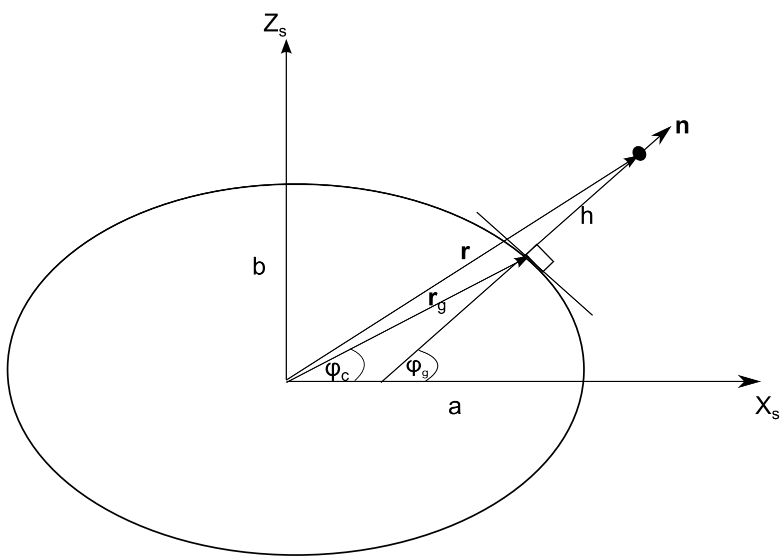

Figure 2: Geodetic vs. Geocentric Latitude

Literature Review

Many methods have been proposed for implementing the inverse of the transformation presented in equation (1). The Cartesian-to-Geodetic transformation problem is challenging because the governing equations are nonlinear. Many solution strategies suffer from geometrical singularities. Two problem areas commonly arise: (1) poor convergence near the Poles, and (2) handling very sensitive quartic polynomial solutions [1, 2, 3]. Many fundamentally different algorithms are able to produce useful results. At the heart of most algorithms is an exploitation of the underlying elliptical geometry of the Earth's surface. Often these approaches involve clever use of trigonometric identifies for either defining and/or simplifying the necessary conditions that define the solution for the inversion process. Surprisingly, very few perturbation-based approaches have been proposed, even though, as shown in this work, a very natural perturbation parameter exists for the Earth's surface elliptical geometry. All recent research has been focused on developing robust and computationally efficient algorithms.

The solution for the geodetic longitude is elementary and non-iterative. The solution for the geodetic latitude and satellite height are coupled and highly nonlinear. Three classes of methods have been proposed (1) closed-form solutions for cubic and quartic polynomials, (2) perturbation methods, and (3) successive approximation algorithms.

Closed-Form Algorithms. The *closed-form* class of solution algorithms typically introduces sequences of trigonometric transformations that exploit identities to simplify the governing equation. Important examples of this approach include: (1) Bowring's [4] very well-known solution, where the reduced latitude is iterated in Newton's method; (2) Vaníčˇek and Krakiwski [5] solve a high-order algebraic equation in closed-form to solve the problem; (3) Pick [6] introduces the geodetic height of the satellite to develop an elliptic integral-based arc-length solution, which is elegant analytically, but computationally inefficient; (4) Fotiou [7] develops a low-order approximate closed-form solution; (5) Vermeille [8] develops a series solution by introducing a complicated algebraic transformation. The proposed closed-form algorithms are extremely accurate; nevertheless, they are often computationally very expensive often because special function calculations are required for which highly optimized software is not available.

Iterative and Perturbation Algorithms. Many *iterative techniques* have been proposed. As with all iterative methods for nonlinear problems, one is always concerned with picking "good" starting guess, which leads to rapid convergence. Early examples of this approach include the work of Heiskanen and Moritz [9] which influenced the GPS-based need for the geodetic transformation methods developed by Kleusber and Teunissen [10], and Hofmann-Wellenhop et al. [11]. Several innovative problem formulations have been proposed, including the work of Torge [12], Borkowski [2], Lin and Wang [13], Fukushima [1], and Lupash [14]. Unfortunately, geometric singularities plague many of these iterative strategies. Most often, the problems encountered arise near the Poles where the math models for the local geometry break down, leading to a severe loss of solution accuracy for the problem variables. Several authors have investigated *vector methods* to avoid troublesome singularities, as captured in the foundational work of Pollard [3] and Feltens [15]. An elegant optimization-based strategy has been presented by Zhang et al. [16], which reformulates the problem as a constrained minimization problem, which requires the extraction of a very sensitive root of a fourth-order polynomial. Shu and Li [17] build on the work of Zhang et al. [16] and present a special starting guess for the governing quartic equation, which yields high accuracy and overcomes the quartic polynomial inversion problems encountered by Fukushima [1], Borkowski [2], Pollard [3], and Lin and Wang [13]. Accelerated convergence techniques are considered by Fukushima [19], who has presented a third-order version of Newton's method (known as Halley's method). Turner [19] has extended the work of Zhang et al. [16] by introducing an artificial perturbation variable that transforms the classical quartic solution problem into a singularity-free non-iterative quadratic equation problem, which is analytically inverted using perturbation methods. He improves the accuracy of the perturbation method by introducing Padé approximations that significantly boost the accuracy of the perturbation solution. Turner [21] formulates the necessary conditions into the form of a rapidly convergent generalized continued faction expansion. A highly accurate starting guess (i.e. ~ 10 digits) is obtained for the expansions by developing a 2^{nd} order approximation for the infinite limit for the convergent of the generalized continued fraction expansion.

The main contribution of this work is the presentation of a perturbation series solution that non-iteratively inverts the Cartesian to Geodetic transformation throughout the LEO-to-GEO range. The solution accuracy renders the solution effectively a closed-form solution. No Polar singularities are encountered. The series solution coefficients are very compact and are highly efficient for computation.

MATH MODELS

The solution for λ_g is obtained by elementary methods, and is given by

$$\lambda_g = \operatorname{atan}\left(\frac{y_s}{x_s}\right) \tag{2}$$

From Eq. (1) it is obvious that the inversion for the remaining variable is fundamentally a nonlinear problem Normally, successive approximation strategies are required for ϕ_g and h. Algorithmically, it is simpler to formulate problem necessary conditions in terms of the geocentric anomaly ϕ_c, rather than the geodetic anomaly ϕ_g appearing in Eq. (1), because: (1) the Earth's elliptical surface is exactly described in terms of the geocentric anomaly, and (2) the normal vector to the Earth's elliptical surface, which points at the satellite, is easily parameterized in terms of the geocentric anomaly. As a result, after inverting the problem for the geocentric anomaly and the satellite height; standard trigonometric identities are used to transform the geocentric anomaly into the desired geodetic anomaly.

Perturbation Series Solution

A successive approximation strategy is developed by introducing a naturally available geometric perturbation variable. Referring to the geometry of the Earth's elliptical surface, one defines the naturally occurring perturbation variable

$$p = a / b - 1 \approx 0.0034 \tag{3}$$

which suggests that a 4th order expansion should be accurate to $\approx 10^{-12}$. Numerical experiments with the series solution confirm this expectation. Rapidly convergent approximations are obtained for ϕ_c and h in the $\hat{\tau} - \hat{k}$ plane by developing power series in the expansion variable p.

Math Model. The problem is formulated by introducing a local coordinate system that tracks the local *x-y* axis motion of the satellite. In the local coordinate system, a simplified perturbation solution is developed in the $\hat{\tau} - \hat{k}$ plane by defining a vector constraint of the form

$$\mathbf{r} - \mathbf{r}_g - h\hat{\mathbf{n}} = \mathbf{0} \tag{4}$$

where $\mathbf{r} = (r_{xy}, z)$ denotes the satellite position and the vector $r_{xy} = \sqrt{x^2 + y^2}$, $\mathbf{r}_g = (a\cos(\phi_c), b\sin(\phi_c))$ denotes the satellite ground track point, ϕ_c denotes the geocentric latitude, h denotes the height of the satellite above the Earth's surface, and

$$\hat{\mathbf{n}} = \left(\frac{\cos(\phi_c)}{a}, \frac{\sin(\phi_c)}{b} \right) \Big/ \sqrt{\left(\left(\frac{\cos(\phi_c)}{a} \right)^2 + \left(\frac{\sin(\phi_c)}{b} \right)^2 \right)}$$

denotes the unit vector that is normal to the Earth's surface and points at the satellite. Expanding Eq. (4) provides two necessary conditions

$$\begin{aligned} r_{xy} - a\cos(\phi_c) - \frac{h\cos(\phi_c)}{a\sqrt{\left(\left(\frac{\cos(\phi_c)}{a} \right)^2 + \left(\frac{\sin(\phi_c)}{b} \right)^2 \right)}} &= 0 \\ z - b\sin(\phi_c) - \frac{h\sin(\phi_c)}{b\sqrt{\left(\left(\frac{\cos(\phi_c)}{a} \right)^2 + \left(\frac{\sin(\phi_c)}{b} \right)^2 \right)}} &= 0 \end{aligned} \tag{5}$$

Clearly the equations are still highly nonlinear. To begin the simplification process one replaces a in Eq. (5) with

$$a = b(1 + p); \quad p \approx 0.00314$$

Leading the transformed equations

$$r_{xy} - b(1+p)\cos(\phi_c) - \frac{h\cos(\phi_c)}{\sqrt{\left((\cos(\phi_c))^2 + ((1+p)\sin(\phi_c))^2\right)}} = 0$$
$$z - b\sin(\phi_c) - \frac{h\sin(\phi_c)}{\sqrt{\left(\left(\frac{\cos(\phi_c)}{1+p}\right)^2 + (\sin(\phi_c))^2\right)}} = 0 \tag{6}$$

These equations are now in a form that can be solved by a perturbation power series solution, by assuming that the geocentric latitude and satellite height are expanded in the fourth-order series representations.

$$\begin{aligned} \phi_c &= \phi_0 + p\phi_1 + p^2\phi_2 + p^3\phi_3 + p^4\phi_4 + \cdots \\ h &= h_0 + ph_1 + p^2h_2 + p^3h_3 + p^4h_4 + \cdots \end{aligned} \tag{7}$$

The series solution coefficients are recovered by introducing Eq. (7) into Eq. (6) and developing the cascade of coefficient necessary conditions, in powers of p, leading to:

$$O(p^0): h_0 = \sqrt{r_{xy}^2 + z^2} - b; \qquad \phi_0 = 2\tan^{-1}\left(\frac{\sqrt{r_{xy}^2 + z^2}}{z} - \frac{r_{xy}}{z}\right)$$

$$O(p^1): h_1 = -b\cos(\phi_0)^2; \qquad \phi_1 = \frac{b-h_0}{2(b+h_0)}\sin(2\phi_0)$$

$$O(p^2): h_2 = \frac{b(3b-h_0)}{8(b+h_0)}\sin(2\phi_0)^2; \ \ \phi_2 = \frac{h_0^2 - 4bh_0 + 3b^2}{8(b+h_0)^2}\sin(4\phi_0) + \frac{\sin(2\phi_0)}{4}$$

$$O(p^3): h_3 = \frac{b\sin^2(2\phi_0)}{8(b+h_0)^2}\left[(h_0 - 3b)^2\cos^2(\phi_0) + 4b(h_0 - b)\right]$$

$$\phi_3 = \frac{\sin(2\phi_0)}{6(b+h_0)^3}\left[C_1\cos^4(\phi_0) + C_2\cos^2(\phi_0) + C_3\right]$$

$$O(p^4): h_4 = \frac{b\sin^2(2\phi_0)}{32(b+h_0)^3}\left[C_4\cos^4(\phi_0) + C_5\cos^2(\phi_0) + C_6\right]$$

$$\phi_4 = \frac{\sin(2\phi_0)}{4(b+h_0)^4}\left[C_7\cos^6(\phi_0) + C_8\cos^4(\phi_0) + C_9\cos^2(\phi_0) + C_{10}\right] \tag{8}$$

where simple algebraic manipulations yield the ten coefficients appearing in Eq. (8) as the polynomials presented in Table 1. The analytic form for the coefficients is remarkably simple, and was obtained using the computer algebra system Masyma 2.4. Many of the terms appearing in Table 1 can be pre-computed to accelerate the calculations. The conversion from the geocentric to the geodetic latitude is given by

$$\phi_g = \tan^{-1}\left(\frac{\cos\phi_c}{a/b\sin\phi_c}\right)$$

Numerical experiments confirm that the 4th order expansion provides effectively a closed-form solution for the entire LEO-to-GEO range of applications.

Table 1. Polynomial Expansion Coefficients

Coefficient	Expression
C_1	$-4h_0^3+37b^3-66b^2h_0+33bh_0^2$
C_2	$h_0^3-31b^3+75b^2h_0-33bh_0^2$
C_3	$3b\left(b^2+3h_0^2-6bh_0\right)$
C_4	$139b^3-5h_0^3+49bh_0^2-143b^2h_0$
C_5	$h_0^3-127b^3+163b^2h_0-45bh_0^2$
C_6	$4b\left(-10bh_0+h_0^2+5b^2\right)$
C_7	$4h_0^4+118b^4+198b^2h_0^2-266b^3h_0-54bh_0^3$
C_8	$-155b^4-2h_0^4+67bh_0^3+421b^3h_0-315b^2h_0^2$
C_9	$49b^4-15bh_0^3-185b^3h_0+135b^3h_0^2$
C_{10}	$2b^2\left(10bh_0-5h_0^2-b^2\right)$

Numerical Experiments

Extensive calculations are performed to assess the accuracy of the proposed perturbation series solution. The perturbation expansion method is used to carry out the coordinate transformation for many cases of LEO to GEO orbits, including the poles. Using the WGS84, the forward transformation is carried out first, then the perturbation solution is applied and the results are compared with the original values, which represents exact values for the inverse problem solution. Separate accuracy results are obtained for 2nd, 3rd, and 4th order approximations. For the sake of demonstration; a longitude angle of 30° is utilized. The geodetic latitude, ϕ_g, is swept for angles between -90 to 90 degrees and the height is swept from 200 KM (LEO) to 35,000 KM (GEO). First the expansion is carried to second order and the errors in latitude and height are plotted as functions of the true latitudes and heights as shown in Figure 3 and Figure 4, respectively. The expansion is then carried out to third order and the errors for latitude and height are plotted in Figure 5 and Figure 6, respectively. Finally, the fourth order expansion is utilized and results are shown in Figure 7 and Figure 8, respectively.

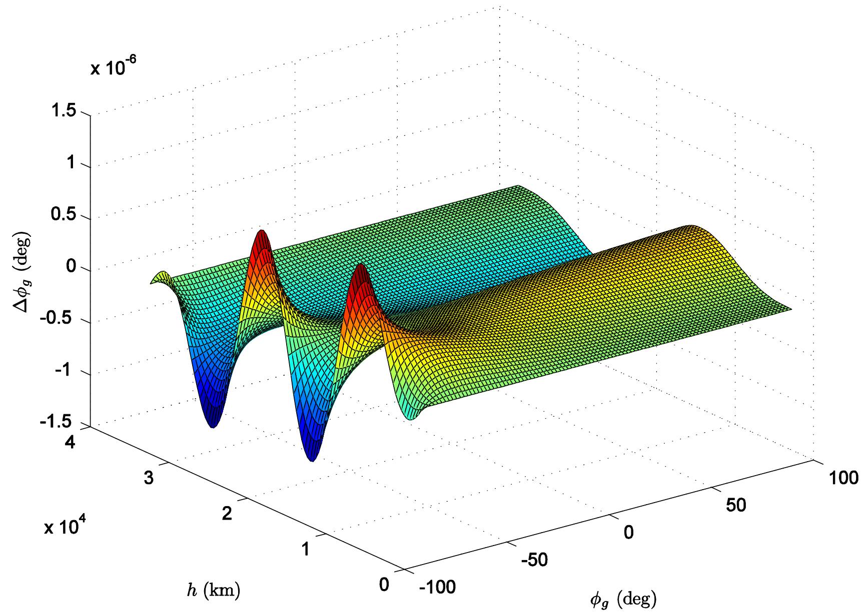

Figure 3. Errors in latitude, 2nd order expansion

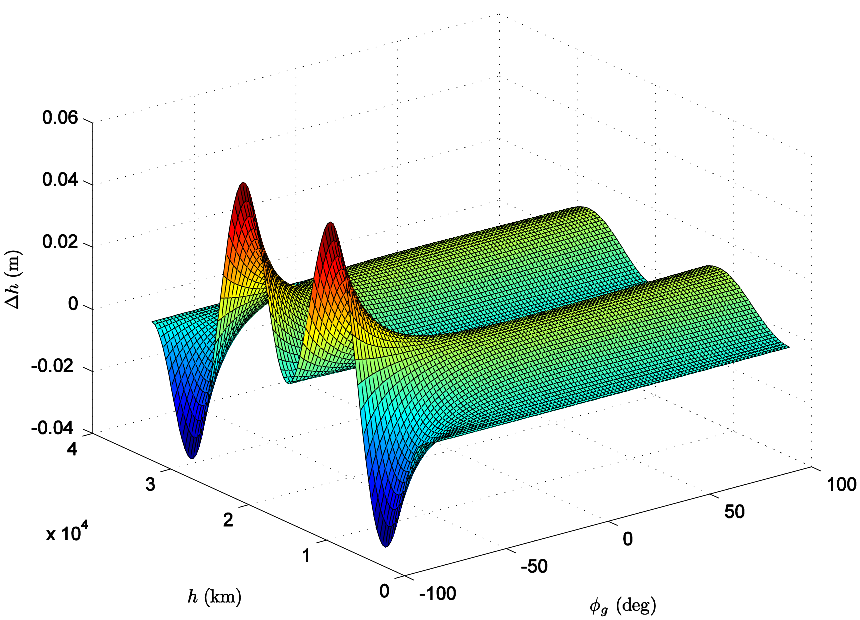

Figure 4. Errors in height, 2nd order expansion

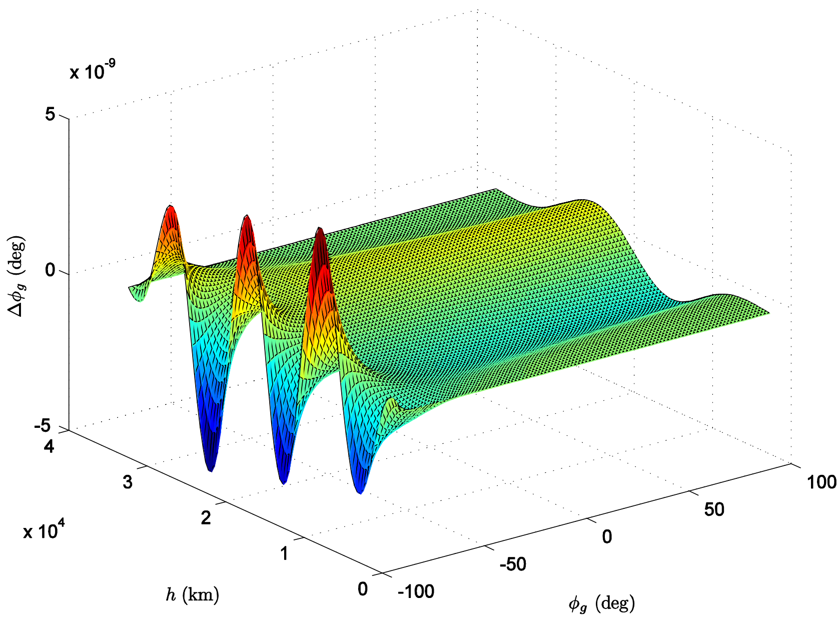

Figure 5. Errors in latitude, 3rd order expansion

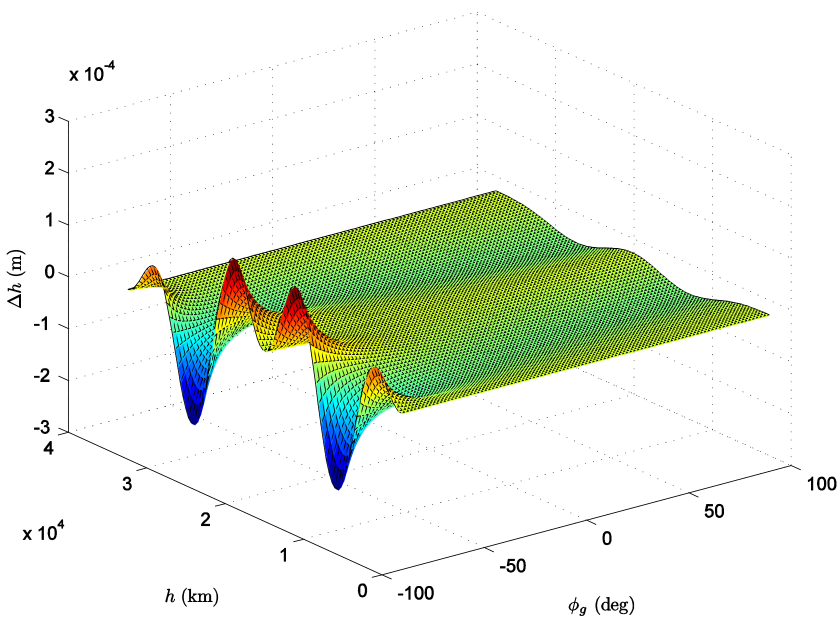

Figure 6. Errors in height, 3rd order expansion

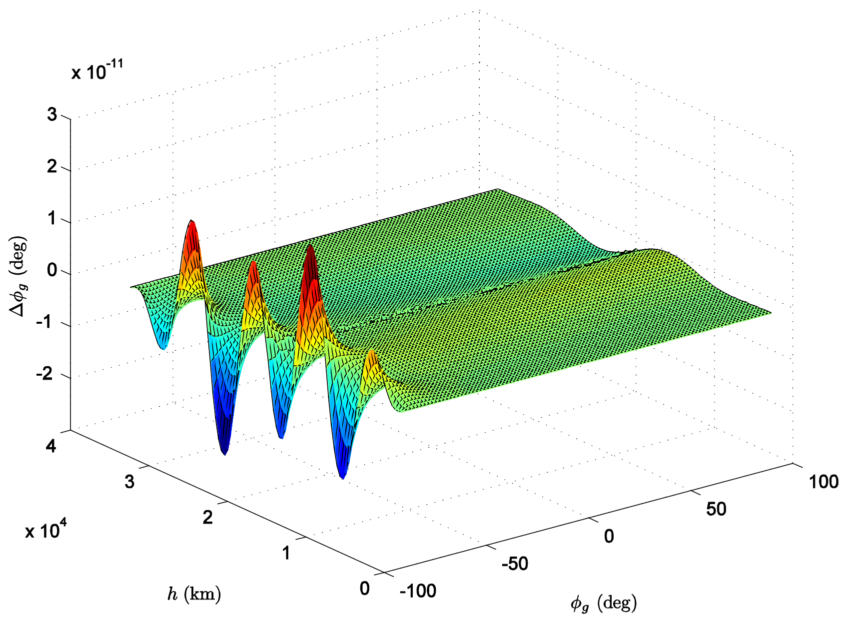

Figure 7. Errors in latitude, 4th order expansion

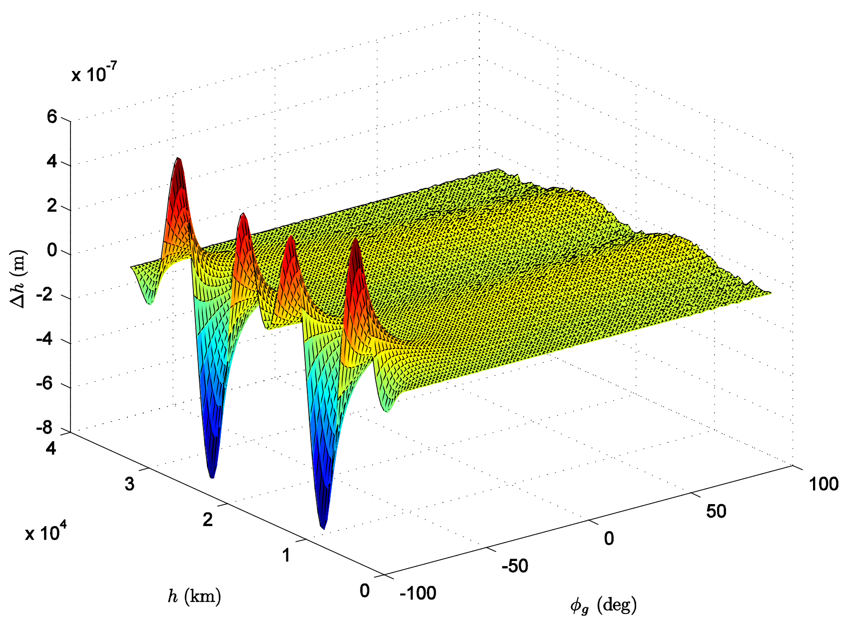

Figure 8. Errors in height, 4th order expansion

The improvement of accuracy is quite obvious as the order of expansion is increased. Two orders of magnitude improvement is achieved by adding the third order terms to each of the coordinates. Another two orders of magnitude improvement is achieved with the fourth order terms. In height mm accuracy is achieved at the fourth order expansion level. This shows the fast convergence nature and the accuracy of the perturbation solution. These results demonstrate that higher-order approximations do not provide additional useful information for the inversion process.

CONCLUSION

Earth Centered Earth Fixed (ECEF) to geodetic coordinate transformation has been examined with several numerical and analytical approaches throughout the literature. A non-iterative expansion based approach inspired by the Earth's perturbed geometry is introduced in this work, where the expansion parameter is nothing but the ratio of the Earth semi-major and minor axis subtracted from 1. The expansion is carried out to second, third and fourth orders. A numerical example is introduced to compare the accuracies at each order of expansion. Accuracies showed significant improvements as the order of expansion is increased and at fourth order mm accuracy is achieved in height and 10^{-11} degrees error in latitude. Those errors at such low orders of the expansion are proof of the effectiveness of the method its potential in solving such a highly nonlinear transformation non-iteratively. The method can be further streamlined for timing studies but in general it is a clean straight forward approach to the coordinate transformation problem that utilizes a physical perturbation parameter and that proved to be very accurate and efficient.

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