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Generalized Frequency Domain State-Space Models for Analyzing Flexible Rotating Spacecraft

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Abstract

The mathematical model for a flexible spacecraft that is rotating about a single axis rotation is described by coupled rigid and flexible body degrees-of-freedom, where the equations of motion are modeled by integro-partial differential equations. Beam-like structures are often useful for analyzing boom-like flexible appendages. The equations of motion are analyzed by introducing generalized Fourier series that transform the governing equations into a system of ordinary differential equations. Though technically straightforward, two problems arise with this approach: (1) the model is frequency-truncated because a finite number of series terms are retained in the model, and (2) computationally intense matrix-valued transfer function calculations are required for understanding the frequency domain behavior of the system. Both of these problems are resolved by: (1) computing the Laplace transform of the governing integro-partial differential equation of motion; and (2) introducing a generalized state space (consisting of the deformational coordinate and three spatial partial derivatives, as well as single and double spatial integrals of the deformational coordinate). The resulting math model is cast in the form of a linear state-space differential equation that is solved in terms of a matrix exponential and convolution integral. The structural boundary conditions defined by Hamilton's principle are enforced on the closed-form solution for the generalized state space. The generalized state space model is then manipulated to provide analytic scalar transfer function models for original integro-partial differential system dynamics. Symbolic methods are used to obtain closed-form eigen decomposition–based solutions for the matrix exponential/convolution integral algorithm. Numerical results are presented that compare the classical series based approach with the generalized state space approach for computing representative spacecraft transfer function models.

Introduction

Simplified models of maneuvering flexible spacecraft are often modeled as coupled rigid hub/beam-like structures. Mathematically, these systems are de-

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scribed by coupled systems of integro-partial differential equations (*IPDE*) [1–9]. Classically generalized Fourier series models are introduced to transform the *IPDE* models into ordinary differential equations. This approach has been very successful. Nevertheless, high-accuracy solution strategies require many series terms, leading to large matrix equations for evaluating the system transfer functions. The need for dealing with large matrix equations is eliminated by introducing a generalized state-space (*GSS*) model that retains an exact s-domain flexible body model for the coupled rigid or flexible body system. Three steps are required for developing a transfer function model for the governing system *IPDE*: (1) the *IPDE* is Laplace transformed to yield a spatial *IPDE*; (2) the integral part of the *IPDE* is simplified by introducing integration-by-parts to yield a generalized integral equation involving multiple integrals of the beam variable; and (3) a *GSS* model is defined that replaces the *IPDE* model with a spatial 6×1 linear matrix-vector differential equation. No frequency truncation is introduced. A closed-form s-domain solution is obtained for the system matrix exponential and convolution integral by invoking symbolic methods. The structural behavior for the vehicle is enforced by imposing the boundary conditions obtained from an application of Hamilton's principle. Four boundary conditions are imposed on the structural response: (1) enforcing the geometric boundary conditions for attaching the beam to the rigid hub (i.e., $y(r) = 0$, $y'(r) = 0$), and (2) enforcing the satisfaction of the physical boundary conditions for the free end of the beam (i.e., $EI\frac{\partial^2 y}{\partial x^2}\Big|_{x=r+L} = 0$, $EI\frac{\partial^3 y}{\partial x^3}\Big|_{x=r+L} = 0$). With the geometric and physical boundary conditions satisfied, one easily obtains scalar transfer function models from the *GSS* model. Numerical results are presented to compare the accuracy and efficiency of both classical and the proposed *GSS* transfer function algorithms.

The major innovation of the article is the introduction of a Laplace transformed-based *GSS* model for analyzing the behavior of an *IPDE* system dynamics model. The *GSS* model consists of the state, several partial derivatives of the state, as well as single and double integrals of the flexible body state variable (i.e., an *IPDE* state space). *GSS* is unconventional because it mixes variables evaluated at points in the structural domain, as well as integrals of the structural response over the entire flexible body domain. No Generalized Fourier Series representations are introduced for modeling the deformational degree-of-freedom (*DOF*). Closed-form solutions are obtained for all state vector elements. The main contributions of this article are: (1) the development of a closed-form Laplace transform-based solution for 6×6 matrix exponential that governs the behavior of a coupled hub/beam system, (2) rigorous scalar-valued distributed parameter transfer function models that are suitable for conducting engineering design iterations, and (3) the *GSS* model has no frequency truncation (it is effectively a closed-form solution for the flexible body behavior).

Article Organization

The article is organized in the following way. The governing equation of motion (*EOM*) is presented in the math modeling section. A classical transform function approach is presented that transforms the *IPDE* model by introducing a generalized Fourier Series-based approach for the rotating rigid hub/beam dynamics problem.

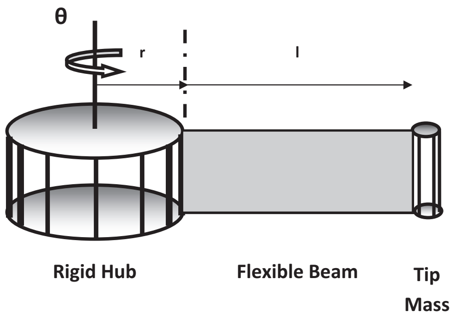

FIG. 1. *GSS* and Classical Transfer Function Evaluated at $x = 1.0$ (Free End).

The *GSS* method is presented that allows the *IPDE* model to be cast in the form of a generalized integral equation that is modeled in terms of a linear spatial state space model. The *GSS* model is analyzed in terms of a matrix exponential and convolution integral representation. Three steps are required for analyzing the *s*-domain *6 x 6* matrix exponential solution. First, the 4×4 beam submatrix of the matrix exponential is solved in closed-form by using a right- and left-eigensolution strategy. Second, the complete 6×6 matrix exponential solution is obtained by using analytic insights gleaned from the 4×4 subproblem solution. Third, the matrix exponential and convolution integral are transformed into an elegant form by introducing a change of variable that exploits a complex variable identity; which permits the physical beam boundary conditions at the free end of the beam to be enforced. The *GSS*-based transfer function section presents closed-form scalar results for various system-level transfer functions. The numerical results section presents several examples of both classical series-based approaches and the corresponding *GSS*-based transfer function results. The impacts of model truncation are examined in this section. The results of the article are summarized in the conclusion section of the article.

Math Model

The coupled hub/beam model, Fig. 1, combines a rotating rigid hub and cantilevered boom as a single subsystem. Only single-axis maneuvers are considered. The rigid hub is assumed to rotate about its local z-axis and the attached boom is allowed to have transverse deformations about the local y-axis. The undeformed boom lies along the local x-axis, where the attached appendage is assumed to be a uniform flexible beam subject to standard Euler–Bernoulli assumptions of negligible shear deformation and negligible distributed rotatory inertia. The *IPDE* is linearized by dropping the velocity component $-y\dot{\theta}$ term. Each substructure is modeled in terms of its kinetic and potential energy. The *EOM* are developed by invoking Hamilton's extended principle [10], yielding a coupled system of *IPDEs*, as well as the required geometric and physical boundary conditions for the beam structure.

The *IPDE* system is Laplace transformed and the resulting equations are manipulated into the form of a linear matrix differential equation; yielding a closed-form s-domain spatial matrix exponential/convolution integral solution. A two-part derivation approach is presented for the s-domain 6×6 matrix exponential: (1) The 4×4 beam subpart of the matrix is solved in closed-form using a symbolic eigensolution method; and (2) The 4×4 beam closed-form solution is used to recover closed-form solutions for the remaining elements of the coupled 6×6 hub/beam *GSS* matrix exponential elements. The symbolically obtained solutions are validated by comparing series-based solutions with Taylor expansions of the closed-form solutions. A symbolic bi-orthogonal eigen decomposition algorithm is applied for the 4×4 beam submatrix part of the matrix exponential [11] that provides an analytic solution. The full matrix exponential is recovered by symbolically Taylor expanding the matrix and comparing results with the beam submatrix at the element-by-element level. The vector elements of the convolution matrix integral are evaluated in closed-form.

Future implementations of these substructure models are expected to produce significant computational reductions in the computational effort required to analyze the frequency domain behaviors for engineering-level-of-fidelity models for rotating rigid bodies with attached beam-like structures.

Kinetic and Potential Energy

The kinetic and potential energies of the coupled rigid hub/beam hybrid system are given as [3, 8, 9]

$$2T = J_h\dot{\theta}^2 + \int_r^{r+l} \left[\rho(\dot{y} + x\dot{\theta})^2\right] \mathrm{d}x; \qquad 2V = \int_r^{r+l} \left[EI(y_{,xx})^2\right] \mathrm{d}x$$

The nonconservative virtual work for this system follows as

$$\delta W_{nc} = u\delta\theta$$

where θ denotes the rigid body rotation angle for central rigid hub, y denotes the transverse beam deflection coordinate, E denotes the elastic modulus of the beam, J denotes the moment of combined inertia for the rigid hub and the undeformed beam, I denotes the moment of inertia for the beam, ρ denotes the linear beam mass density, A denotes the beam cross sectional area, and u denotes the torque applied to the rigid hub.

Rotating Linked Hub and Flexing Beam Math Model

Application of Hamilton's principle [10] for the model presented in Fig. 1 leads to the *EOM* and governing geometric or physical boundary conditions:

Integro-Partial Differential Equation of Motion Model (IPDE). The math model for the vehicle consists of coupled angular momentum and elastic beam equations, as

$$J\ddot{\theta} + \int_r^{r+l} \underbrace{\rho A x \ddot{y} \mathrm{d}x}_{\substack{\textit{Bending Ietegral}\\ \textit{Coupling}}} = u \qquad J = \underbrace{J_h}_{\substack{\textit{Hub}\\ \textit{Inertia}}} + \underbrace{\int_r^{r+l} \rho A x^2 \mathrm{d}x}_{\substack{\textit{Undeformed}\\ \textit{Beam Inertia}}} \tag{1}$$

TABLE 1. Geometric and Physical Boundary Conditions

Boundary Conditions	Model
Displacement	$y\|_r = 0$
Slope	$y_{,x}\|_r = 0$
Bending moment	$EI\dfrac{\partial^2 y}{\partial x^2}\Big\|_{x=r+L} = 0$
Shear force	$EI\dfrac{\partial^3 y}{\partial x^3}\Big\|_{x=r+L} = 0$

$$\rho A\left(\ddot{y} + \underbrace{x\ddot{\theta}}_{\substack{Rigid\\Body\\Coupling}}\right) + EIy_{,xxxx} = 0 \tag{2}$$

where the explicit x-axis dependence complicates the subsequent analysis. Hamilton's principle provides the geometric and physical boundary conditions listed in Table 1.

Equations (1) and (2) are hard to solve because both time and space variables appear. Classically this problem is handled by introducing a generalized Fourier series, where displacement shapes are provided for eliminating the spatial dependence after integration over the spatial domain. A linear matrix-vector system of ordinary differential equations is obtained for the coupled hub/flexible beam system. The model is frequency truncated because only a finite number of terms are retained in the generalized Fourier series representation. Model truncation carries with it the risk that disturbance sources with significant energy in the range of the truncated frequencies can lead to poor structural response predictions, Fig. 2. Model truncation is eliminated in this article by combining the Laplace transform method and a *GSS*.

Classical Series Expansion Approach for Modeling Hub/Beam Structures. A brief review of the classical transfer function approach follows to facilitate the comparison of classical and *GSS*-based Transfer function methods. The structural response is analyzed by assuming a Generalized Fourier series [10, 12] as

$$y(x,t) = \sum_{r=1}^{N} \phi_r(x)\,\eta_r(t) = \underbrace{\Phi^{\mathrm{T}}(x)}_{1xN}\underbrace{\Upsilon(t)}_{Nx1}$$

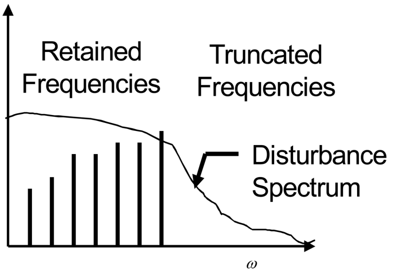

FIG. 2. Hub, Beam and Tip Mass Distributed Parameter System.

where $\phi_r(x)$ denotes the r-th flexural displacement shape and $\eta_t(t)$ denotes the modal amplitude for the flexural displacement shape. Assuming the power series model, equations (1) and (2) become

$$J\ddot{\theta} + \left(\int_r^{r+l} \rho A x \Phi^{\mathrm{T}}(x)\mathrm{d}x\right)\ddot{\Upsilon} = u$$

$$\rho A \Phi^{\mathrm{T}}(x)\ddot{\Upsilon} + \rho A x\, \ddot{\theta} + EI\Phi^{\mathrm{T}}_{,xxxx}(x)\Upsilon = 0$$

The spatial dependence is eliminated from the second equation by premultiplying by the displacement vector and integrating over the domain of interest, leading to the transformed equations

$$J\ddot{\theta} + \left(\int_r^{r+l} \rho A x \Phi^{\mathrm{T}}(x)\mathrm{d}x\right)\ddot{\Upsilon} = u$$

$$\left(\int_r^{r+l} \rho A \Phi(x)\Phi^{\mathrm{T}}(x)\mathrm{d}x\right)\ddot{\Upsilon} + \left(\int_r^{r+l} \rho A x \Phi(x)\mathrm{d}x\right)\ddot{\theta} + \left(EI\int_r^{r+l} \Phi(x)\Phi^{\mathrm{T}}_{,xxxx}(x)\mathrm{d}x\right)\Upsilon = 0$$

or

$$M\ddot{x} + Kx = \mathbf{f} \tag{3}$$

where

$$M = \begin{bmatrix} J & M_\theta^{\mathrm{T}} \\ M_\theta & M_{\phi\phi} \end{bmatrix} \qquad K = \begin{bmatrix} 0 & 0^{\mathrm{T}} \\ 0 & K_{\phi\phi} \end{bmatrix} \qquad M_\theta = \int_r^{r+l} \rho A x \Phi(x)\mathrm{d}x$$

$$M_{\phi\phi} = \int_r^{r+l} \rho A \Phi(x)\Phi^{\mathrm{T}}(x)\mathrm{d}x \qquad K_{\phi\phi} = EI\int_r^{r+l} \Phi(x)\Phi^{\mathrm{T}}_{,xxxx}(x)\mathrm{d}x$$

Laplace Transform Model. The integral definition of the Laplace transform is given by

$$\bar{f}(\mu) = \int_0^\infty e^{-st}f(t)\,\mathrm{d}t$$

that when applied to equation (3) yields the transfer function

$$s^2 M\bar{\mathbf{x}} + K\bar{\mathbf{x}} = \bar{\mathbf{f}} \qquad \bar{\mathbf{x}} = [s^2M + K]^{-1}\bar{\mathbf{f}} \tag{4}$$

The computational cost associated with evaluating the transfer function is dominated by the symmetric matrix inversion calculation that scales as n^3, where n denotes the number of modes retained in the model. A symmetric matrix inversion is computed for each value of frequency. Alternatively, the *IPDE* transfer function calculation presented in this article replaces the expensive frequency truncated matrix inversion calculation with a scalar inversion calculation where all structural frequencies are retained.

The Laplace Transform of the *EOM* defined by equations (1) and (2), yields the coupled pair of *spatial IPDE*'s

$$s^2 J\bar{\theta} + s^2 \rho A \int_r^{r+l} x\,\bar{y}\mathrm{d}x = \bar{u} \tag{5}$$

$$EI\,\bar{y}_{,xxxx} + s^2 \rho A(\bar{y} + x\bar{\theta}) = 0$$

The development of a *GSS* model is simplified by replacing the integral term with the equivalent *integration-by-parts* solution

$$\int_r^{r+l} x\bar{y}\mathrm{d}x = x\int_r^{r+l} \bar{y}\mathrm{d}x - \int_r\int \bar{y}\mathrm{d}x\mathrm{d}x'$$

that replaces the integrations with integrands only involving the Laplace transform of the flexible body distributed parameter variable. Introducing the integration-by-parts solution into equation (5) yields the desired form for the *spatial IPDE EOM*

$$s^2 J\bar{\theta} + s^2 \rho A\left(x\int_r^{r+l} \bar{y}\mathrm{d}x - \int_r\int \bar{y}\mathrm{d}x\mathrm{d}x'\right) = \bar{u} \tag{6}$$

$$EI\,\bar{y}_{,xxxx} + s^2 \rho A(\bar{y} + x\bar{\theta}) = 0$$

that is now classified as a generalized integral equation because of the appearance of the double integral. The second equation is an ordinary fourth order differential equation. Both equations must be solved simultaneously. A standard state space approach is not equal to the task: a new state space is required that simultaneously handles all of the flexible body terms, including: $\bar{y}_{xxxx}$, $\bar{y}$, $\int \bar{y}\mathrm{d}x$, and $\int\int \bar{y}\mathrm{d}x\mathrm{d}x'$.

Spatial Integro-Differential State Space Model

Six states are required because four partial derivatives and two integrals must be handled for the flexible body variable $\bar{y}(x)$. $\bar{\theta}$ is not included in the *GSS* because it has no spatial dependence. To this end, defining the spatial derivative as $(*)' = \partial(*)/\partial x$, the state variables for the *GSS* follow as

$$
\begin{aligned}
z_1 &= \iint \bar{y}\,\mathrm{d}x\mathrm{d}x' & z_1' &= z_2 \\
z_2 &= \int \bar{y}\,\mathrm{d}x & z_2' &= z_3 \\
z_3 &= \bar{y} & z_3' &= z_4 \\
z_4 &= \bar{y}' & z_4' &= z_5 \\
z_5 &= \bar{y}'' & z_5' &= z_6
\end{aligned} \tag{7}
$$

$$
z_6 = \bar{y}''' \qquad z_6' = -\beta(z_3 + x\bar{\theta}) \qquad \beta = \frac{s^2\rho A}{EI}
$$

The state space is generalized in the sense that both partial derivatives and integral terms appear explicitly in the variable definitions. The rigid body motion appears as a forcing function. All variables appear linearly, allowing closed-form solutions to be obtained for the matrix exponential-based algorithm.

Boundary-Value Solution for the GSS Initial Conditions. The *GSS* initial conditions are given by

$$
\mathbf{Z}\big|_{x=0} = [\,z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5 \quad z_6\,]^{\mathrm{T}}\big|_{x=0}
$$

The first two terms are zero because the integrals are assumed to be zero initially. The third and fourth terms vanish because of the geometrical boundary conditions defined by Table 1, leading to

$$
\mathbf{Z}\big|_{x=0} = [0 \quad 0 \quad 0 \quad 0 \quad z_5 \quad z_6]^{\mathrm{T}}\big|_{x=0} \tag{8}
$$

The remaining solutions for z_5 and z_6 are defined by enforcing the physical boundary conditions defining force and moment balance conditions at the end of the beam (see Table 1). These boundary conditions are evaluated after the complete spatial *IPDE* solution has been analytically integrated.

Using the variable definitions in equation (7) the rigid body equation of equation (6) is expressed as

$$
s^2 J\bar{\theta} + s^2\rho A(x\,z_2 - z_1) = \bar{u} \tag{9}
$$

where the second term on the left provides an exact representation for the distributed parameter beam coupling effects for the rigid-hub/beam system.

s-Domain Spatial Linear Matrix Differential Equation. Equation (7) is recast as the following 6×6 linear matrix differential equation

$$
\mathbf{Z}' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\beta & 0 & 0 & 0 \end{bmatrix} \mathbf{Z} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\beta x\bar{\theta} \end{Bmatrix} = [A]\mathbf{Z} + \mathbf{b} \tag{10}
$$

which has the well-known solution [8, 11]

$$\mathbf{Z}\big|_{x=l} = \exp[Al]\mathbf{Z}\big|_{x=0} + \int \exp[A(l-\tau)]b(\tau)\mathrm{d}\tau \tag{11}$$

The matrix exponential of equation (11) is easily evaluated by developing a series expansion as

$$\exp[Al] = I + Al + \frac{1}{2}(Al)^2 + \frac{1}{6}(Al)^3 + \cdots$$

Nevertheless, when many values of μ must be evaluated, computational efficiency becomes a serious issue. To reduce the computational impact of repeated matrix evaluations, we seek a closed-form solution for the matrix exponential. To this end, a two-step approach is presented: (1) the beam 4×4 subproblem is solved using a symbolic eigen decomposition technique, and (2) the 6×6 *GSS* solution is obtained using analytical insights gained from the 4×4 beam subproblem solution. This strategy decouples the flexible body part of the solution from the extended calculation involving first and second integrals of the flexible body response.

<u>Closed-Form Solution for the Spatial State Matrix Exponential</u>. The Laplace transform for the beam part of the EOM is defined as

$$EI\,\bar{y}_{,xxxx} + s^2\rho A(\bar{y} + x\bar{\theta}) = 0$$

where rigid body coupling term $s^2\rho Ax\bar{\theta}$ acts as a forcing term. In terms of the *GSS* defined by equation (7), the unforced beam subproblem consists of the 4×4 beam bending linear matrix differential equation given by

$$Q' = CQ \qquad Q = \exp[Cx]Q\big|_{x=0} \tag{12}$$

where

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

The solution for equation (12) is obtained from the eigen decomposition of C in terms of its right- and left-eigenvectors. The eigen decomposition for C is obtained symbolically by using the computer aided algebra program MACSYMA. The resulting analytic solution is shown to exhibit group-like properties for the elements of the solution. The closed-form solutions obtained for equation (12) are validated by Taylor expanding the analytic solutions and comparing the results with the series expansions for the matrix exponentials. The insights gleaned from solving equation (12) guide the solution strategy for obtaining a closed-form solution for the 6×6 system appearing in equation (10).

Beam Subproblem Matrix Exponential

A closed-form solution is obtained for $Q' = CQ$ by exploiting the bi-orthogonality conditions for the eigenvectors for C [8, 11]. The *right*- and *left-eigenvectors* are obtained by solving the eigenvalue problems

$$
\begin{aligned}
CR_i &= \lambda_i R_i, \quad i = 1,2,\cdots, n \\
C^t L_j &= \lambda_j L_j, \quad i = 1,2,\cdots, n \\
L_j^t R_i &= \delta_{ij}, \quad i,j = 1,2,\cdots, n
\end{aligned}
\tag{14}
$$

The 4×4 diagonal matrix containing the eigenvalues for C is given by

$$D = \beta^{1/4}\,\mathrm{Diag}\left[-\frac{1}{\sqrt{i}} \quad -\sqrt{i} \quad \frac{1}{\sqrt{i}} \quad \sqrt{i} \right]$$

where $i = \sqrt{-1}$ denotes an imaginary complex number. With R, L, and D known, one can express the C matrix in terms of its eigen decomposition. Observe that the eigenvectors are normalized by $L^{\mathrm{T}}R = I \Rightarrow R^{-1} = L^{\mathrm{T}}$. From equation (14) it follows that one can write $CR = RD$. Postmultiplying this result by R^{-1} leads to desired result

$$C = RDR^{-1} = RDL^{\mathrm{T}}$$

Introducing C into the matrix exponential and factoring out the *right-* and *left-eigenvectors*, yields

$$\exp[Cx] = \exp[RDL^{\mathrm{T}}x] = \mathrm{R}\exp[Dx]\,L^{\mathrm{T}} \tag{15}$$

where

$$\exp[Dx] = \begin{bmatrix} e^{-\beta^{1/4}x/\sqrt{i}} & 0 & 0 & 0 \\ 0 & e^{-\sqrt{i}\beta^{1/4}x} & 0 & 0 \\ 0 & 0 & e^{\beta^{1/4}x/\sqrt{i}} & 0 \\ 0 & 0 & 0 & e^{\sqrt{i}\beta^{1/4}x} \end{bmatrix}$$

A complete expansion of equation (15) has been obtained by using the computer aided algebra program MACSYMA 2.4. The final results are not presented here because of space limitations. A careful examination of the symbolic results provided for equation (15) indicates that the matrix exponential encodes the repetitive substructure

$$\exp[Cx] = \begin{bmatrix} f & -f'''/\beta & -f''/\beta & -f'/\beta \\ f' & f & -f'''/\beta & -f''/\beta \\ f'' & f' & f & -f'''/\beta \\ f''' & f'' & f' & f \end{bmatrix} \tag{16}$$

where the following matrix element exists along the diagonal

$$f(x) = \cos\left(\frac{\beta^{1/4}x}{\sqrt{2}}\right)\cosh\left(\frac{\beta^{1/4}x}{\sqrt{2}}\right) \tag{17}$$

The identification of the matrix structure in equation (16) in terms of the derivatives of f provides the critical insight required for completely solving the 6×6 GSS matrix exponential of equation (10). Equations (15) and (16) are checked by symbolically evaluating the following expansion

$$I + Cx + C^2x^2/2! + \cdots + C^Nx^N/N! \approx \exp[Cx]$$

GSS Matrix Exponential Solution

Comparing Taylor expansions for exp $[Ax]$ and exp $[Cx]$, one easily establishes data structure for exp $[Ax]$ as

$$\exp[Ax] = \begin{bmatrix} 1 & x & -f''/\beta & -f'/\beta & (1-f)/\beta & (\beta x + f''')/\beta^2 \\ 0 & 1 & -f'''/\beta & -f''/\beta & -f'/\beta & (1-f)/\beta \\ 0 & 0 & f & -f'''/\beta & -f''/\beta & -f'/\beta \\ 0 & 0 & f' & f & -f'''/\beta & -f''/\beta \\ 0 & 0 & f'' & f' & f & -f'''/\beta \\ 0 & 0 & f''' & f'' & f' & f \end{bmatrix} \tag{18}$$

where the 4×4 beam substructure matrix exponential is preserved.

Inverse Matrix Exponential Solution

The inverse of the matrix exponential appearing in equation (18) is required for the forced convolution part of the closed-form solution provided by equation (11). By a similar process one can establish that the inverse matrix exponential is given by

$$\exp[-Ax] = \begin{bmatrix} 1 & -x & -f''/\beta & f'/\beta & (1-f)/\beta & -(\beta x + f''')/\beta^2 \\ 0 & 1 & f'''/\beta & -f''/\beta & f'/\beta & (1-f)/\beta \\ 0 & 0 & f & f'''/\beta & -f''/\beta & f'/\beta \\ 0 & 0 & -f' & f & f'''/\beta & -f''/\beta \\ 0 & 0 & f'' & -f' & f & f'''/\beta \\ 0 & 0 & -f''' & f'' & -f' & f \end{bmatrix} \tag{19}$$

This equation has been checked by introducing equation (17) into equations (18) and (19), multiplying the results, and applying trig identities to confirm that the product is a 6×6 identity matrix.

Complex Form of GSS Matrix Exponential

A very compact form of equations (18) and (19) is obtained by recognizing that f in equation (17) represents the *real part* of the complex function

$$f = \cos(\sigma x) \tag{20}$$

where

$$\sigma = \beta^{1/4}(1+i)/\sqrt{2} = \sqrt{i\sqrt{\beta}} \tag{21}$$

To this end, one can express the matrix exponential function of equation (18) in the transformed compact form

$$\exp[Ax] = \begin{bmatrix} 1 & x & \sigma^2 c/\beta & \sigma s/\beta & (1-c)/\beta & (\sigma^3 s + \beta x)/\beta^2 \\ 0 & 1 & -\sigma^3 s/\beta & \sigma^2 c/\beta & \sigma s/\beta & (1-c)/\beta \\ 0 & 0 & c & -\sigma^3 s/\beta & \sigma^2 c/\beta & \sigma s/\beta \\ 0 & 0 & -\sigma s & c & -\sigma^3 s/\beta & \sigma^2 c/\beta \\ 0 & 0 & -\sigma^2 c & -\sigma s & c & -\sigma^3 s/\beta \\ 0 & 0 & \sigma^3 s & -\sigma^2 c & -\sigma s & c \end{bmatrix} \tag{22}$$

where $s = \sin(\sigma x)$ and $c = \cos(\sigma x)$. It is indeed remarkable that the coupled rigid hub/flexible beam *IPDE* is mathematically described by such a simple matrix exponential. Equation (22) has been validated by (1) carrying out Taylor expansions of equation (22), (2) collecting the real part, and (3) expanding exp[Ax] to the same order, and comparing individual terms. Not surprisingly this simple change of variables also greatly simplifies the convolution integral calculations for the forced part of the solution for equation (10).

Closed-Form Solution Components

The solution for *GSS* is completed by introducing equation (22) into

$$Z(x) = \exp[Ax]\, Z_0 + \int_0^x \exp[A(x-\tau)]\; b(\tau)\mathrm{d}\tau \tag{23}$$

and recalling the initial condition vector defined by equation (8). The *homogenous term* is straightforward and can be shown to be

$$\mathbf{Z}_H(x) = \begin{bmatrix} (1-c(\sigma x)z_5/\beta + (\sigma^3 s(\sigma x)+\beta x)z_6/\beta^2 \\ \sigma s(\sigma x)z_5/\beta + (1-c(\sigma x))z_6/\beta \\ \sigma^2 c(\sigma x)z_5/\beta + \sigma s(\sigma x)z_6/\beta \\ -\sigma^3 s(\sigma x)z_5/\beta + \sigma^2 c(\sigma x)z_6/\beta \\ c(\sigma x)z_5 - \sigma^3 s(\sigma x)z_6/\beta \\ -\sigma s(\sigma x)z_5 + c(\sigma x)z_6 \end{bmatrix} \tag{24}$$

The forced solution second term becomes

$$\int_0^x \exp[A(x-\tau)]\, b(\tau)\mathrm{d}\tau = \int_0^x \exp[A(x-\tau)]\,(0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -\beta\tau\bar{\theta})^{\mathrm{T}}\mathrm{d}\tau$$

that reduces to

$$\int_0^x \exp[A(x-\tau)]\, b(\tau)\mathrm{d}\tau = -\beta\bar{\theta}\int_0^x \tau \begin{pmatrix} (\beta(x-\tau)+f'''(x-\tau))/\beta^2 \\ (1-f(x-\tau))/\beta \\ -f'(x-\tau)/\beta \\ -f''(x-\tau)/\beta \\ -f'''(x-\tau)/\beta \\ f(x-\tau) \end{pmatrix} \mathrm{d}\tau \tag{25}$$

Introducing the complex variable form for f defined by equation (20), and integrating the vector set of terms defined by equation (25), leads to

$$\int_0^x \exp[A(x-\tau)]\, b(\tau)\mathrm{d}\tau = (\, I_1 \quad I_2 \quad I_3 \quad I_4 \quad I_5 \quad I_6 \,)^{\mathrm{T}}$$

where the individual convolution integrals can be shown to be

$$\begin{gathered} I_1 = -\frac{\bar{\theta}\left(-\sigma\, sin\,(\sigma x) + \dfrac{\beta x^3}{6} + \sigma^2 x\right)}{\beta} \quad I_2 = -\frac{\bar{\theta}}{\sigma^2}\left(\cos(\sigma x) - 1 + \frac{\sigma^2 x^2}{2}\right) \\ I_3 = -\sigma\bar{\theta}\left(\frac{x}{\sigma} - \frac{\sin(\sigma x)}{\sigma^2}\right) \quad I_4 = -\bar{\theta}(1-\cos(\sigma x)) \\ I_5 = \bar{\theta}(\sigma^2 x - \sigma sin\,(\sigma x)) \quad I_6 = \frac{\beta\bar{\theta}}{\sigma^2}(1-\cos(\sigma x)) \end{gathered} \tag{26}$$

Terminal Physical Boundary Conditions for the Beam Bending Solution

The homogeneous and forced parts of the spatial solution are collected as

$$\mathbf{Z}(x) = \begin{pmatrix} (1 - c(\sigma x)z_5/\beta + (\sigma^3 s(\sigma x) + \beta x)z_6/\beta^2 + I_1(x) \\ \sigma s(\sigma x)z_5/\beta + (1 - c(\sigma x))z_6/\beta + I_2(x) \\ \sigma^2 c(\sigma x)z_5/\beta + \sigma s(\sigma x)z_6/\beta + I_3(x) \\ -\sigma^3 s(\sigma x)z_5/\beta + \sigma^2 c(\sigma x)z_6/\beta + I_4(x) \\ c(\sigma x)z_5 - \sigma^3 s(\sigma x)z_6/\beta + I_5(x) \\ -\sigma s(\sigma x)z_5 + c(\sigma x)z_6 + I_6(x) \end{pmatrix} \tag{27}$$

The only unknowns are the boundary conditions for z_5 and z_6 which are recovered by enforcing the moment and shear conditions at the end of the beam. The last two equations of equation (27) have the information required for solving for the unknowns, leading to the necessary condition for the physical boundary conditions given by

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{bmatrix} c(\sigma L) & -\sigma^3 s(\sigma L)/\beta \\ -\sigma s(\sigma L) & c(\sigma L) \end{bmatrix} \begin{pmatrix} z_5 \\ z_6 \end{pmatrix} + \begin{pmatrix} I_5(L) \\ I_6(L) \end{pmatrix}$$

that is analytically inverted for the initial condition parameters, yielding

$$\begin{pmatrix} z_5 \\ z_6 \end{pmatrix} = \frac{\beta\bar{\theta}}{c(\sigma L)^2 - \sigma^4 s(\sigma L)^2/\beta} \begin{bmatrix} c(\sigma L) & \sigma^3 s(\sigma L)/\beta \\ \sigma s(\sigma L) & c(\sigma L) \end{bmatrix} \begin{pmatrix} I_5(L) \\ I_6(L) \end{pmatrix} \tag{28}$$

Introducing the convolution integrals defined by equation (26), completes the solution for the initial coefficients, leading to

$$z_5 = -\frac{\beta\sigma(s(\sigma L) - \sigma L c(\sigma L))\bar{\theta}}{\sigma^4 s^2(\sigma L) - \beta c^2(\sigma L)}$$

$$z_6 = -\frac{\beta(\sigma^4 s(\sigma L)(s(\sigma L) - \sigma L) - \beta^2 c(\sigma L)(c(\sigma L) + 1))\bar{\theta}}{\sigma^2(\sigma^4 s^2(\sigma L) - \beta c^2(\sigma L))}$$

The full *GSS* solution process is completed by introducing these equations into equation (27), and substituting the complex parameter for σ defined by equation (21) into the resulting equation. The desired solution is recovered by evaluating the real part of the equations as

$$Y(x) = \Re e(Z(x))\bar{\theta} \tag{29}$$

where $\bar{\theta}$ is factored out of the vector of solutions. This complicated step is preformed symbolically. Equation (29) provides the response solution required for the transfer function calculations.

Transfer Function Calculations

Transfer function calculations are presented for rotational and flexible body coupling effects.

TABLE 2. Numerical Values for Model Parameters

Parameter	Description	Value
m_h	Mass of the hub	16
m_a	Mass of the arm/beam	0.10875
D_h	Diameter of hub	2
L	Length of the arm/beam	1
E	Bending Stiffness of arm	1
I_h	Hub moment of inertia	1
ρ	Mass density of arm	1
A	Cross sectional area of arm	1
J	Total inertia	$J = I_h + \frac{\rho A L^3}{3} = 1.33$

Rotational Transfer Function. Introducing the initial condition solution provided by equation (24) into equation (8) and the result into equation (9) yields the single and double integral solutions

$$
\begin{aligned}
Z_1 &= ((1-c)z_5 + (\sigma^3 s + \beta x)z_6)/\beta \\
&= \frac{\bar{\theta}}{c^2 - \sigma^4 s^2/\beta}\begin{pmatrix} 1-c \\ \sigma^3 s + \beta x \end{pmatrix}^T \begin{bmatrix} c & \sigma^3 s/\beta \\ \sigma s & c \end{bmatrix}\begin{pmatrix} I_5(L) \\ I_6(L) \end{pmatrix} \\
&= \bar{\theta} g_5(s)
\end{aligned}
$$

$$
\begin{aligned}
Z_2 &= \frac{\bar{\theta}}{c^2 - \sigma^4 s^2/\beta}\begin{pmatrix} \sigma s \\ 1-c \end{pmatrix}^T \begin{bmatrix} c & \sigma^3 s/\beta \\ \sigma s & c \end{bmatrix}\begin{pmatrix} I_5(L) \\ I_6(L) \end{pmatrix} \\
&= \bar{\theta} g_6(s)
\end{aligned}
$$

Introducing the integral terms in the angular momentum equation leads to

$$
s^2(J + \rho A(xZ_2(x,s) - Z_1(x,s)))\bar{\theta} = \bar{u} \tag{30}
$$

that is inverted for the hub rotational transfer function given by

$$
\bar{\theta} = \frac{\bar{u}}{s^2(J + \rho A(xg_5(x,s) - g_6(x,s)))} \tag{31}
$$

Flexible Body Transfer Function. The flexible body element is defined by the third element the *GSS* state as

$$
\begin{aligned}
\bar{y} &= (\sigma^2 c z_5 + \sigma s z_6)/\beta \qquad (32) \\
&= \frac{\bar{\theta}}{c^2 - \sigma^4 s^2/\beta}\begin{pmatrix} \sigma^2 c \\ \sigma s \end{pmatrix}^T \begin{bmatrix} c & \sigma^3 s/\beta \\ \sigma s & c \end{bmatrix}\begin{pmatrix} I_5(L) \\ I_6(L) \end{pmatrix} \\
&= \bar{\theta} g_3(s) \\
&= \frac{g_3(x,s)\bar{u}}{s^2(J + \rho A(xg_5(x,s) - g_6(x,s)))}
\end{aligned}
$$

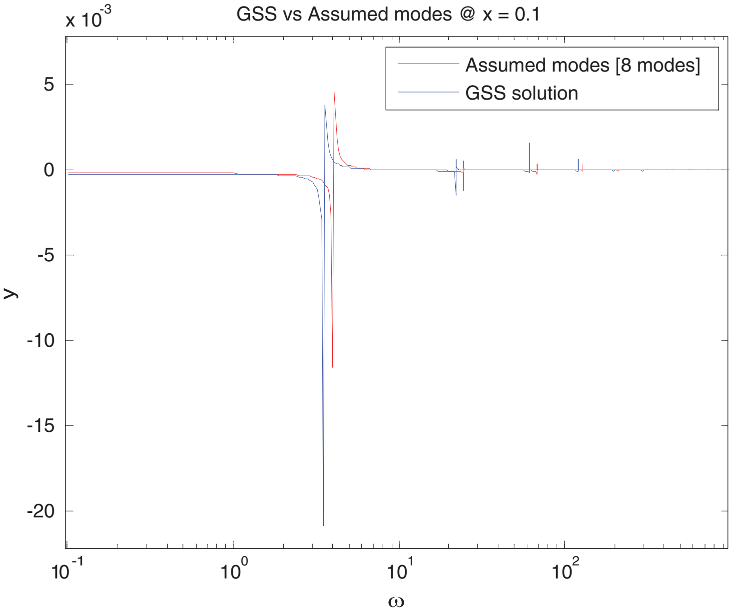

FIG. 3. Illustration of Moel Truncation.

It is obvious that equations (31) and (32) are scalar equations and easily computed. This is in stark contrast to the situation where series approximations are used and the respective transfer functions require the numerical inversion of high-order matrix inversion algorithms.

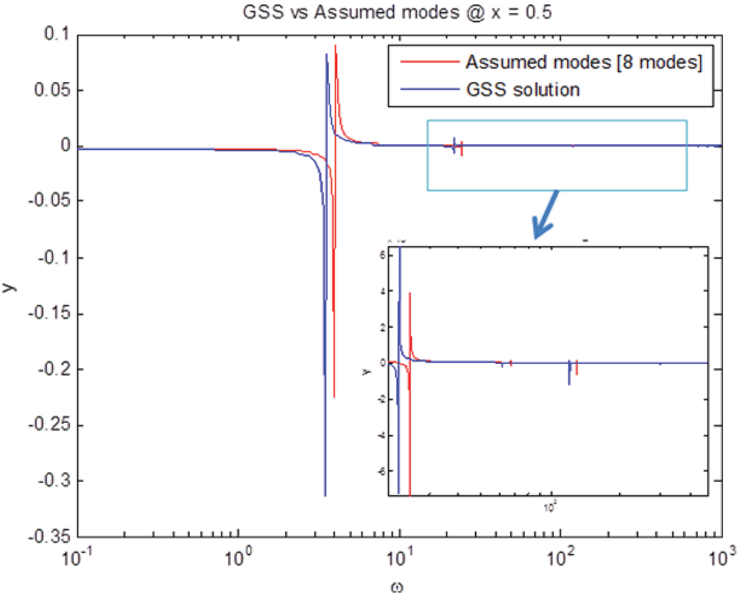

FIG. 4. *GSS* and Classical Transfer Function Evaluated at $x = 0.1$.

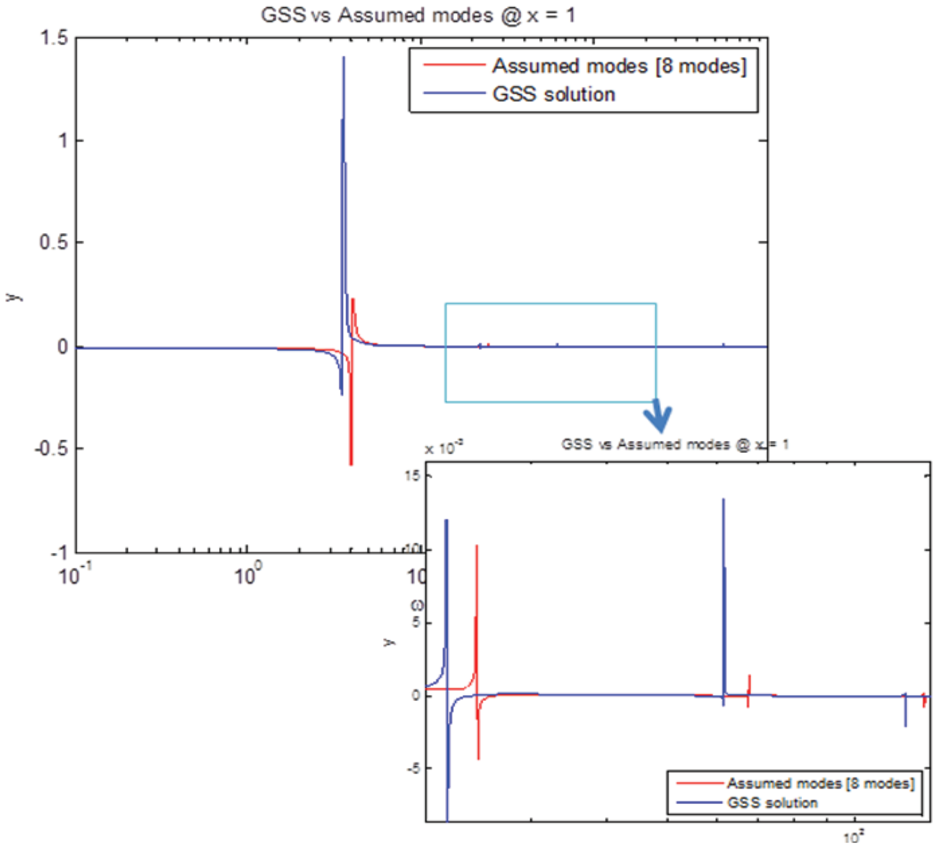

FIG. 5. *GSS* and Classical Transfer Function Evaluated at $x = 0.5$.

Numerical Results

The hub/beam model parameters are presented in Table 2, which have been selected for demonstration purposes only; the parameters do not represent a physical structure and the numbers are assumed to be nondimensional.

The matrix-valued transfer function of equation (4) is evaluated by defining an assumed shape for the deformation behavior for the beam. Numerical results are reported for an eight-mode model where the assumed mode is defined by [11]

$$\phi(x) = 1 - \cos\left(\frac{p\pi x}{L}\right) + \frac{1}{2}(-1)^{p+1}\frac{p\pi x^2}{L}; \qquad p = 1,2,\cdots,8 \tag{33}$$

Equation (33) satisfies the geometric boundary conditions at the attachment point between the hub and the flexible beam. The moment boundary condition at the free end of the beam is not satisfied, however, the shear boundary condition is satisfied. The structural integrals are evaluated symbolically and the matrices are built and evaluated using Matlab. The *GSS* transfer functions are processed symbolically and numerically evaluated. Several plots are presented that depict the recovered transfer functions as a function of frequency for several points along the beam. Figure 3 presents the transfer function evaluated at x = 0.1. In every case the assumed modes frequency estimates converge from above as the number of

TABLE 3. Summary of Numerical Experiments Results

Hub Inertia	Frequencies (rad/s)				
0	15.4	50.0	104.2	178	272
0.005	14.3	37.6	69.5	125	202
0.01	13.3	31.5	65.3	123	201
1	4.01	22.2	61.7	121	200
8	3.50	22.0	61.7	121	200

modes increase. As the frequency increases one can observe that the error in the frequency becomes more pronounced.

In Figs. 4 and 5 as one moves further away from the hub the *GSS* model captures a greater structural response. The beam tip response is very small. Even though the eight modes are used for the assumed modes method it is clear that significant errors exist in the transfer function response predictions, which is important to understand for control strategies and the sensitivities of these algorithms to plant errors in the frequency estimates.

A second numerical experiment is presented, where the hub inertia is varied from zero to large values to sweep the range of beam behaviors from free-free boundary conditions to cantilever boundary conditions. The results of these experiments are summarized in Table 3, where the beam frequencies are seen to rapidly decrease as the hub inertia in increased from zero to a large value.

Conclusions

A symbolically derived analytic solution is presented for a hybrid dynamical system consisting of a rigid hub with an attached flexible appendage. The analytic solution approach is compared with the traditional computationally intense problem arising in spacecraft applications, where high-order series approximations are introduced. Closed-form solutions are obtained for the Laplace transforms for the integro-partial differential equation of motion. A generalized state space is introduced that combines the state, partial derivatives, and integral variables. The introduction of a generalized state space is seen to enable the evaluation of the *s*-domain spatial response in closed-form where each variable is analytically described by scalar variables. The spatial linear matrix exponential solution for the *GSS* is shown to have a very simple structure that is expected to provide critical insights for generalizing the current problem formulation to handle hub/beam translation, symmetric and antisymmetric deformational shapes, distributed control, active structures, material damping, wave motion behaviors, and control approaches, as well as extensions that include beam torsion and higher-order models.

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