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TRANSIENT CONTROL IN MULTIVARIABLE SYSTEMS: A STUDY MOTIVATED BY FUEL CELLS

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ABSTRACT

Controlling the transient response of variables for which sensing or accurate estimation is not feasible, and a detailed plant model is also largely unavailable, poses significant challenges. It is a situation that is true in solid oxide fuel cells. In SOFCs, transient control is essential for fuel utilization, especially if the fuel cell is to be operated in a dynamic load-following mode at high fuel utilization. The objective is to design the control input(s) such that it isolates the output (fuel utilization in this case) from measurable disturbances, while the plant itself maybe largely unknown. The features assumed known are the output's functional dependence on states which is essentially the output definition, and the steady-state equation relating the multiple inputs and the output of interest. Simulations have shown good disturbance rejection in fuel utilization through input shaping. This idea is abstracted to linear multi-variable systems to provide conditions when this approach is applicable. The analysis is carried out in time-domain as well as in frequency domain (through singular value analysis). The type of output variables that are amenable to transient control using this approach is derived through analysis. It is shown that the fuel utilization, although inherently nonlinear within the nonlinear dynamics of the fuel cell, has some similarities with the linear abstraction that leads to the observed transient control.

NOMENCLATURE

F Faraday's constant ($= 96485.34$ coulomb/mol)
i Current draw (A)
k Anode recirculation fraction
N Number of moles (moles)
$\mathcal{N}_{cell}$ Number of cells in series
$\dot{N}_{air}$ Molar flow rate of air (moles/sec)
$\dot{N}_f$ Molar flow rate of fuel (moles/sec)
$\dot{N}_{in}$ Anode inlet flow rate (moles/sec)
$\dot{N}_o$ Anode exit flow rate (moles/sec)
n Number of electrons participating in electrochemical reaction ($= 2$)
P Pressure (N/m^2)
p Partial pressure (N/m^2)
$\mathcal{R}$ Species rate of formation (moles/sec)
R_u Universal Gas Constant (8.314 J/mol/K)
T Temperature (K)
V Volume (m^3)
V_{cell} Cell voltage (V)
$\mathcal{X}$ Species mole fraction

Subscripts

a Anode control volume
c Cathode control volume
i Values of 1 through 7 represent CH_4, CO, CO_2, H_2, H_2O, N_2, and O_2
r Reformate control volume
ss Steady State
fc Fuel cell

INTRODUCTION

Amongst different types of existing fuel cells, SOFCs have advantageous characteristics including high tolerance in the exposure to the impurities, fuel flexibility, and simplicity in com-

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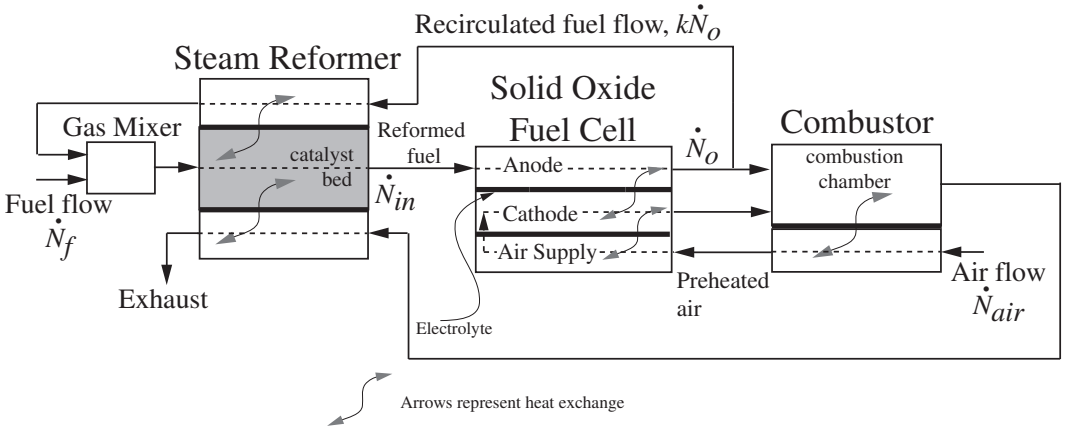

Figure 1. Schematic Diagram of SOFC System

parison with its counterparts [1, 2]. However, some disadvantages such as poor load following capabilities, and instability under transient conditions have limited the application of SOFCs [3]. Deficient load following is primarily due to slow dynamic response of fuel and air delivery devices namely valves, pumps, reformers [4].

One important variable that determines the performance of an SOFC is Fuel Utilization, U, which is defined by the rate of hydrogen consumed over the total hydrogen available. Having done analysis, typical range for U was determined to be $80-85$ percent for optimal efficiency [5, 6]. While utilization below this range implies under power performance and waste of fuel, going far beyond this range leads into fuel starvation and cell degradation, catalyst corrosion, drastic voltage drop [5, 7, 8]. Therefore, an appropriate control scheme is the one which maintains U in the aforementioned range under any transient circumstances. Hence, it is desired to control U within the optimal range (80-90%)even under significant transient circumstances. This target can be challenging as measurement of U is not much accessible since it is a function of molar fractions and flow rates of different species at various locations. Moreover, under large power fluctuations, different types of lags including fuel delivery and recirculation lags could be problematic [7].

In this paper, we focus on the transient control of U as a theoretical problem. We show through both analysis and simulations that a generalized abstraction of the transient control problem in SOFCs is possible. The main idea is that in a multi-input single output system, under certain conditions, it is possible to achieve complete isolation of the output from exogenous input(s) by shaping control inputs, even if the plant is largely unknown or partially known and the output variable itself is not measurable. Such a generalized problem can be posed for non-linear time-varying systems and also for linear time invariant systems. For the latter category of systems, we derive analytical conditions under which the problem is solvable. We also show how these conditions are somewhat satisfied by the fuel cell system, and consequently yield acceptable transient control of U even though the SOFC system is non-linear and time-varying.

BACKGROUND

The reason we are focusing on this type of transient control problem is that in many cases, it is needed to design the control inputs such that the output is independent of one of the exogenous inputs. Specifically, this is the case for SOFCs in which the actual fuel flow rate is governed by power demand, which is an exogenous input. Therefore, we want to design the control input, namely the current draw from the fuel cell, in a way that U undergoes minimum transient fluctuations in the presence of load transients. One could argue that this could be done by sensing the molar fractions of different species in different locations inside the cell; in other words, measuring U at any instant and then designing the control accordingly. Granted this option is possible that requires too much intrinsic sensing within the cell which can be very expensive as well as difficult to implement. Furthermore, estimating U is another possibility through which a control scheme can be developed. Yet, that requires a comprehensive model and state estimator performance would be largely governed by the accuracy of the plant model, which is very approximate in this case. Hence, the challenge and also the novelty of this work is to do the transient control of the output while no instantaneous information of the system is provided but the steady state information.

To demonstrate the SOFC's vulnerability to current draw without proper control, a simulation result is depicted in Fig. 2. Here, we have considered a tubular SOFC model, with 50 cells connected in series, each cell having a length of $251cm$,fuel flow of 7×10^{-4}moles/s and a current draw of 10A for $t<150s$. Using these values, U will be approximately 85%. It is shown that fuel cell is not even able to endure a 1A increase in current load due to hydrogen starvation. This result shows the importance of a proper control as fuel cell performance can be easily disturbed facing small perturbations [9].

System Description

In this work, we consider a tubular SOFC which is essentially constructed by three major parts i.e. the steam reformer,

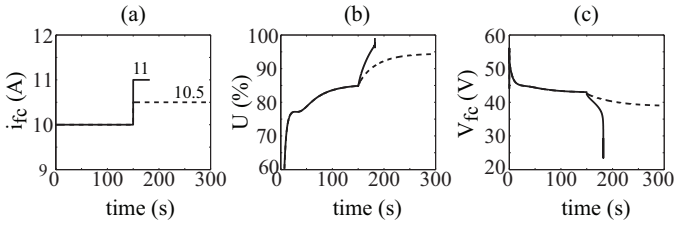

Figure 2. Open-loop Response to Transient Current Demand [9]

the fuel cell stack and the combustor. In the model presented here, primary fuel is chosen to be Methane with a molar flow rate of $\dot{N}_f$. However, the methodology and approach used here can be employed for other fuels and system configurations. The system is illustrated in Fig.1.

The reformer produces a hydrogen-rich gas which is supplied to the anode of the fuel cell. Electrochemical reactions occurring at the anode due to current draw results in a steam-rich gas mixture at its exit. A known fraction k of the anode exhaust is recirculated through the reformer into a mixing chamber where fuel is added. The mixing of the two fluid streams and pressurization is achieved in the gas mixer using an ejector or a recirculating fuel pump [10]. The steam reforming process occurring in the reformer catalyst bed is an endothermic process. The energy required to sustain the process is supplied from two sources, namely, the combustor exhaust that is passed through the reformer, and the aforementioned recirculated anode flow, as shown in Fig.1. The remaining anode exhaust is mixed with the cathode exhaust in the combustion chamber. The combustor also serves to preheat the cathode air which has a molar flow rate of $\dot{N}_{air}$. The tubular construction of each cell causes the air to first enter the cell through the air supply tube and then reverse its direction to enter the cathode chamber. For steam reforming of methane we consider a packed-bed tubular reformer with nickel-alumina catalyst. The three main reactions in steam reforming of methane are [11]:

$$\begin{array}{l}\text{(I)}\quad CH_4+H_2O \leftrightarrow CO+3H_2 \quad \text{(II)}\ CO+H_2O \leftrightarrow CO_2+H_2 \\ \text{(III)}\ CH_4+2H_2O \leftrightarrow CO_2+4H_2\end{array} \tag{1}$$

Internal reforming reactions I, II and III in Eq.(1) occur in the anode due to high temperatures and the presence of nickel catalyst. The primary electrochemical process is:

$$\text{(IV)}\quad H_2+O^{2-} \rightarrow H_2O+2e \tag{2}$$

Details of the system model are presented in [1]. Fuel utilization U is mathematically defined as, [5, 6, 8]:

$$U \triangleq 1-\frac{\dot{N}_o\left(4\mathcal{X}_{1,a}+\mathcal{X}_{2,a}+\mathcal{X}_{4,a}\right)}{\dot{N}_{in}\left(4\mathcal{X}_{1,r}+\mathcal{X}_{2,r}+\mathcal{X}_{4,r}\right)} \tag{3}$$

where, $\mathcal{X}_{1,a}$, $\mathcal{X}_{2,a}$, $\mathcal{X}_{4,a}$ and $\mathcal{X}_{1,r}$, $\mathcal{X}_{2,r}$, $\mathcal{X}_{4,r}$ are the molar concentrations of CH_4, CO and H_2 in the anode and the reformer respectively and $\dot{N}_o$ and $\dot{N}_{in}$ are shown in Fig.1. Eq.(3) is based on the observation that a CH_4 and a CO molecule can yield at most four molecules and one molecule of H_2 respectively, as indicated by reactions I, II and III in Eq.(2).

Existence and Derivation of Invariant Properties

The molar balance equations of individual species in the reformer and anode are:

$$\begin{array}{l}\frac{d}{dt}\left(N_r\mathcal{X}_{1,r}\right)=k\dot{N}_o\mathcal{X}_{1,a}-\dot{N}_{in}\mathcal{X}_{1,r}+\mathcal{R}_{1,r}+\dot{N}_f \\ \frac{d}{dt}\left(N_r\mathcal{X}_{2,r}\right)=k\dot{N}_o\mathcal{X}_{2,a}-\dot{N}_{in}\mathcal{X}_{2,r}+\mathcal{R}_{2,r} \\ \frac{d}{dt}\left(N_r\mathcal{X}_{3,r}\right)=k\dot{N}_o\mathcal{X}_{3,a}-\dot{N}_{in}\mathcal{X}_{3,r}-\mathcal{R}_{1,r}-\mathcal{R}_{2,r} \\ \frac{d}{dt}\left(N_r\mathcal{X}_{4,r}\right)=k\dot{N}_o\mathcal{X}_{4,a}-\dot{N}_{in}\mathcal{X}_{4,r}-4\mathcal{R}_{1,r}-\mathcal{R}_{2,r} \\ \frac{d}{dt}\left(N_r\mathcal{X}_{5,r}\right)=k\dot{N}_o\mathcal{X}_{5,a}-\dot{N}_{in}\mathcal{X}_{5,r}+2\mathcal{R}_{1,r}+\mathcal{R}_{2,r}\end{array}$$

$$\begin{array}{l}\frac{d}{dt}\left(N_a\mathcal{X}_{1,a}\right)=\dot{N}_{in}\mathcal{X}_{1,r}-\dot{N}_o\mathcal{X}_{1,a}+\mathcal{R}_{1,a} \\ \frac{d}{dt}\left(N_a\mathcal{X}_{2,a}\right)=\dot{N}_{in}\mathcal{X}_{2,r}-\dot{N}_o\mathcal{X}_{2,a}+\mathcal{R}_{2,a} \\ \frac{d}{dt}\left(N_a\mathcal{X}_{3,a}\right)=\dot{N}_{in}\mathcal{X}_{3,r}-\dot{N}_o\mathcal{X}_{3,a}-\mathcal{R}_{1,a}-\mathcal{R}_{2,a} \\ \frac{d}{dt}\left(N_a\mathcal{X}_{4,a}\right)=\dot{N}_{in}\mathcal{X}_{4,r}-\dot{N}_o\mathcal{X}_{4,a}-4\mathcal{R}_{1,a}-\mathcal{R}_{2,a}-r_e \\ \frac{d}{dt}\left(N_a\mathcal{X}_{5,a}\right)=\dot{N}_{in}\mathcal{X}_{5,r}-\dot{N}_o\mathcal{X}_{5,a}+2\mathcal{R}_{1,a}+\mathcal{R}_{2,a}+r_e\end{array} \tag{4}$$

$$r_e = i_{fc}\mathcal{N}_{cell}/nF$$

where, r_e is the rate of electrochemical reaction. In Eq(4), $\mathcal{X}_{i,r}$ and $\mathcal{X}_{i,a}$ are the molar concentrations of species in the reformer and anode respectively, with $i=1,2,\cdots,5$ representing CH_4, CO, CO_2, H_2 and H_2O in that order. N_r and N_a are the molar contents of the reformer and the anode, and k is the constant and known recirculation fraction shown in Fig.1. $\mathcal{R}_{1,r}$, $\mathcal{R}_{2,r}$ and $\mathcal{R}_{1,a}$, $\mathcal{R}_{2,a}$ are the rates of formation of CH_4 and CO in the reformer and anode respectively. In Eq.(4), i_{fc} is the fuel cell current, $\mathcal{N}_{cell}$ is number of series-connected cells, $n=2$ is the number of electrons participating in an electrochemical reaction, and $F=96485.34$Coul./mole is the Faraday's constant. Further details about the equations can be found in [1, 12]. From Eqs.(3) and (4), the steady-state utilization U_{ss} is obtained as

$$U_{ss} = (1-k)/\left[\left(4nF\dot{N}_f/i_{fc}\mathcal{N}_{cell}\right)-k\right] \tag{5}$$

Note that Eq.(5) is independent of the reaction rates $\mathcal{R}_{1,r}$, $\mathcal{R}_{2,r}$, $\mathcal{R}_{1,a}$ $\mathcal{R}_{2,a}$ and the flow rates $\dot{N}_{in}$, $\dot{N}_o$. Equation (5) is valid in steady-state and is invariant with respect to variations in operating temperature, operating pressure, mass of reforming catalyst, air flow rate and operating Steam-to-Carbon ratio [1]). Thus, Eq.(5) represents *an invariant relationship* between steady-state fuel utilization U_{ss}, fuel cell current i_{fc}, and fuel flow rate $\dot{N}_f$. Given a target U_{ss}, it can be used to determine $\dot{N}_f$ if i_{fc} is known and vice-versa.

Matrix Representation of Invariant property

This task will involve determining the conditions under which such an invariant property exists. In this regard, some critical observations have been made. For instance,

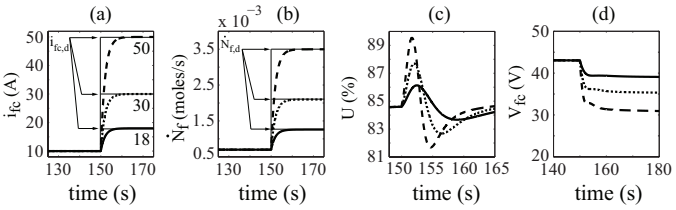

Figure 3. Effect of i_{fc} Regulation on Transient U and V_{fc}

$$U = 1 - \dot{N}_o P^T X_a / \dot{N}_{in} P^T X_r, \quad \begin{matrix} X_a^T = \begin{bmatrix} X_{1,a} & X_{2,a} & X_{3,a} & X_{4,a} & X_{5,a} \end{bmatrix} \\ X_r^T = \begin{bmatrix} X_{1,r} & X_{2,r} & X_{3,r} & X_{4,r} & X_{5,r} \end{bmatrix} \\ P^T = \begin{bmatrix} 4 & 1 & 0 & 1 & 0 \end{bmatrix} \end{matrix} \tag{6}$$

Where X_a and X_r are molar concentration vectors in the anode and reformer respectively. P is also the vector of potential hydrogen content of species. Moreover,

$$P^T M R = 0, \quad M^T = \begin{bmatrix} -1 & 1 & 0 & 3 & -1 \\ -1 & 0 & 1 & 4 & -2 \\ 0 & -1 & 1 & 1 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} r_{I,r} \\ r_{II,r} \\ r_{III,r} \end{bmatrix} \; or \; \begin{bmatrix} r_{I,a} \\ r_{II,a} \\ r_{III,a} \end{bmatrix} \tag{7}$$

Where R is the reaction rate vector and M is reaction matrix whose elements are the coefficients of species in the reforming reactions in Eq.(1).

Note that P lies in the null-space of M, i.e. irrespective of the reaction rates in R, $P^T M = 0$. This orthogonality property of P has led to the invariance of U_{ss} with respect to reaction rates. It has further allowed us to express the steady-state internal hydrogen availabilities given by $\dot{N}_o P^T X_a$ and $\dot{N}_{in} P^T X_r$ in terms of the inputs i_{fc} and $\dot{N}_f$.

Derivation of Invariant Property

The possibility of model-independent derivation of the invariant property arises from the physical interpretations of the terms $\dot{N}_{in} P^T X_r$ and $\dot{N}_o P^T X_a$ appearing in the expression of U in Eq.(6). $\dot{N}_{in} P^T X_r$ and $\dot{N}_o P^T X_a$ represent the net molar hydrogen available in flows going into and out of the SOFC anode respectively and includes the hydrogen that can be generated through SR in addition to pure hydrogen.

$$\dot{N}_{in} P^T X_r - \dot{N}_o P^T X_a = r_e = i_{fc} \mathcal{N}_{cell} / nF \tag{8}$$

Deriving the corresponding mass conservation equation for the reformer in a model-independent manner is more complicated due to the presence of oxidation reactions and since the hydrogen available in fuel must be incorporated. Nonetheless, for the configuration in Fig.1,

$$\dot{N}_{in} P^T X_r - \dot{N}_o P^T X_a = 4\dot{N}_f \tag{9}$$

is independent of internal reaction rates in the reformer.

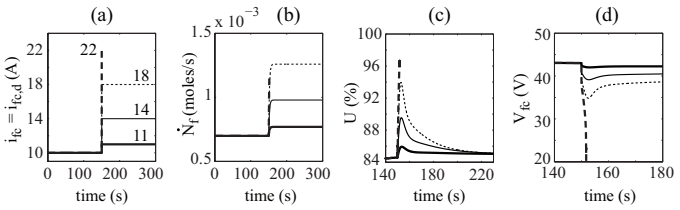

Figure 4. Open-Loop Control of Fuel Cell System using Invariant Property [9]

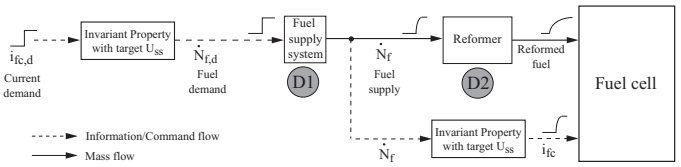

Figure 5. The Concept of Current Shaping

ADDRESSING TRANSIENT PERFORMANCE USING INVARIANT PROPERTY

The invariant property can serve as an open-loop control for constant utilization operation. However, it is a steady-state property and hydrogen starvation must be prevented even during transient operation. Typically, U must be around an optimal value ($\approx 85\%$) within narrow limits ($\pm 5\%$) even under significant power fluctuations. The limitation of using the invariant property alone in an open-loop control is demonstrated in the following simulations of System shown in Fig.1, where instantaneous changes in power demand are applied through step-changes in current demand $i_{fc,d}$. The fuel demand is computed according to the systems invariant property in Eq.(5),

$$\dot{N_{f,d}} = i_{fc,d} \mathcal{N}_{cell} [1 - (1 - U_{ss})k] / 4nFU_{ss} \tag{10}$$

The results, shown in Fig.4, depict significant and prolonged transient departures from target $U_{ss} = 85\%$. Hydrogen starvation, around $i_{fc,d} = 22A$, is manifested by a loss of voltage.

Transient Control through Current Shaping

The principle of this method depicted in Fig.5 is, the fuel cell current i_{fc} is shaped according to a measure of hydrogen availability at the anode inlet. Due to sensing complexity, this is not directly available. Hence, the measured fuel flow from the fuel supply system to the reformer is used for shaping i_{fc}. This is facilitated by the existence of the invariant property relating i_{fc} and $\dot{N}_f$ for a target utilization U_{ss}.

Simulations are presented in Fig.3 to demonstrate the effect of current regulation on transient utilization and voltage. The system simulated is the same as that in Fig.2. Referring to Fig.4, the simulation results represent the current regulation (CR) mode. Three simulations are presented, with target $U_{ss} = 85\%$, $i_{fc} = 10$A for $t < 150$s, and $i_{fc,d} = 18, 30, 50$A for $t \geq 150$s. The resulting transient U and V_{fc} are shown in Figs.3(c) and (d).

A GENERALIZED TRANSIENT CONTROL PROBLEM

From the above outlined transient control problem of Solid Oxide Fuel Cells, a generalized transient control problem can be formulated for multi-input single output (MISO) systems. This generalized problem statement is as follows: Consider a nonlinear and non-autonomous MISO system described as,

$$\dot{X} = f(X,t,u,w), \quad y = h(X,t,u,w) \tag{11}$$

where, X is the state vector, y is the output, u is the vector of control inputs and w is the vector of disturbance/exogenoues inputs. Let us assume that at steady-state,

$$y_{ss} = g(u_{ss}, w_{ss}) \tag{12}$$

where the subscript ss denotes steady-state. Further, let us assume that the following are true:

1. The functions $h(X,t,u,w)$ and $g(u_{ss},w_{ss})$ are known but knowledge of $f(X,t,u,w)$ is "incomplete".
   Note: The exact implication of "incomplete" knowledge of the plant will be clarified in the problem statements in the following sections.
2. X and y are not measurable, but the exogeneous inputs w are measurable.

Then, under what additional condition(s), can we design the control inputs $u = q(w)$ such that $\lim_{t\to\infty} y(t) = \delta$, where δ is a desired/target value of y?

The above problem corresponds closely to the fuel utilization control problem in SOFCs, where X corresponds to the states $P^T X_a$ and $P^T X_r$, given in Eq.(6) and y corresponds to U, whose functional form is known as well. Also, $\dot{N}_{in}$, $\dot{N}_o$, N_r, N_a, are functions of time, making the system non-autonomous. $\dot{N}_f$ is the exogeneous input and i_{fc}, which is shaped based on $\dot{N}_f$ using Eq.(10), is the control input. Eq.(5) corresponds to the function $g(u_{ss},w_{ss})$, assumed known. Also, δ corresponds to the target U_{ss}. We first formulate the problem for a linear time invariant(LTI) MISO system.

PROBLEM FORMULATION FOR AN LTI-MISO SYSTEM

Consider the following LTI-MISO system

$$\dot{X} = AX + B(u + w), \quad y = CX \tag{13}$$

where, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{1\times n}$. Also let $u \in \mathbb{R}^{n\times 1}$ and $w \in \mathbb{R}^{n\times 1}$ represent the vectors containing the control inputs and disturbance inputs respectively. Following to Eq.(13), the steady-state input/output relation is therefore

$$\begin{aligned} &\dot{X_{ss}} = 0 = AX_{ss} + B(u_{ss} + w_{ss}) \\ &\Rightarrow X_{ss} = -A^{-1}B(u_{ss} + w_{ss}) \\ &\Rightarrow y_{ss} = CX_{ss} = -CA^{-1}B(u_{ss} + w_{ss}) \end{aligned} \tag{14}$$

Next, we make the following assumptions that are parallel to those for the generalized nonlinear time-varying case:

1. B and C matrices are known but A is unknown.
2. A is Hurwitz which can be inferred based on an overall physical knowledge of the system.
3. B is invertible.
4. There are $m < n$ disturbance inputs in w and $(n-m) \geq 1$ control inputs in u. This implies that the sum of L_0 "norms" of u and w is n, which essentially means a total of n inputs.
5. $CA^{-1}B$ is known.
   Note: Knowing C, B and Eq.(14) does not imply the knowledge of A.
6. X and y are not measurable, but the exogenous inputs w are measurable.

Note that the set of assumptions above now clearly defines what we mean by incomplete knowledge of $f(X,t,u,w)$ in assumption 1 of the generalized case, but this definition is only applicable for LTI-MISO systems. The question we ask is, can we design $u = q(w)$ such that $\lim_{t\to\infty} y(t) = \delta$ where δ is a desired/target value? To this end, we give the following theorem:

THEOREM 1: *Consider the LTI system given in Eq.(13) for which all the above assumptions are valid. If y is such that CA^{-1} and C are linearly dependent, then by using Eq.(14) with*

$$\delta = -CA^{-1}B(u + w) \tag{15}$$

we can completely eliminate the effect of w on y and ensure $\lim_{t\to\infty} y(t) = \delta$ for any transients in w.

Proof: First, we observe that since C, B and $CA^{-1}B$ are known and B is invertible, CA^{-1} can be determined from the given information. Hence we can verify if C and CA^{-1} are linearly dependent, without knowing A. If they are, then

$$CA^{-1} = \alpha C, \quad \alpha < 0 \tag{16}$$

Here $\alpha < 0$ because Eq.(16) implies

$$C = \alpha CA \;\Rightarrow\; C\left(\frac{1}{\alpha}I - A\right) = 0 \tag{17}$$

and hence $1/\alpha$ is an eigenvalue of A and C is the corresponding eigenvector. Since A is known to be Hurwitz, α is negative real. Now, taking the derivative of Y, we have

$$\dot{y} = CAX + CB(u + w) \;\Rightarrow\; \dot{y} = \frac{1}{\alpha}y + CB(u + w) \tag{18}$$

which implies that by choosing u to satisfy

$$CB(u + w) = CBu + CBw = -\frac{\delta}{\alpha} \;\Rightarrow \lim_{t\to\infty} y(t) = \delta \tag{19}$$

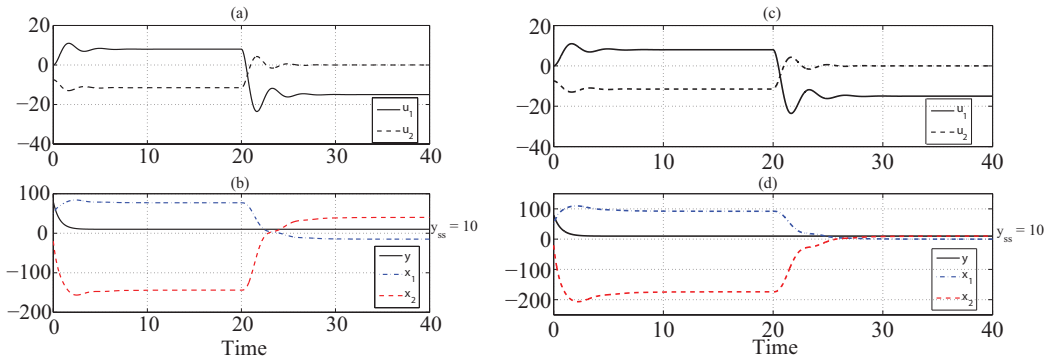

Figure 6. Simulation Showing Isolation Of y From Exogenous Input u_1 During Transients.

From Eq.(16), $CA^{-1} = \alpha C$ and therefore by choosing u to satisfy

$$CA^{-1}B(u + w) = -\delta \tag{20}$$

we ensure $\lim_{t\to\infty} y(t) = \delta$. This completes the proof. $\diamond\diamond\diamond$

We note that we are essentially using the steady-state equation Eq.(14) to *shape* u using the measurements of w to ensure complete disturbance rejection in y. Also, given that w is measurable, we only need one control input to control the transients in y. Any extra control input can be used to achieve other control objectives. We also make the following observations:

Remark 1: *Given a MISO-LTI system with B, C and $CA^{-1}B$ satisfying all the conditions of Theorem 1, there exists a continuum of realizations of admissible A matrices.*

Remark 2: *A loss of observability is an enabler for transient control of y, since linear dependence of C and CA^{-1} implies a rank 1 observability matrix.*

To illustrate the effectiveness of the Theorem 1, simulation results is are shown in Fig.6(a),(b). The system chosen in 6(a) is

$$A = \begin{bmatrix} 1 & 0.75 \\ -5 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 \end{bmatrix} \tag{21}$$

in which u_1 is considered an exogenous input and u_2 is the control input. Thus, $u = [u_1 \ 0]^T$ and $w = [0 \ u_2]^T$. The matrices B, C and $CA^{-1}B = [2/3 \ 4/3]$ are assumed known. A is Hurwitz (eigenvalues at -0.5 and -1.5) but is assumed unknown. It can be verified that B is invertible. Using B and $CA^{-1}B$, we can verify that $CA^{-1} = -(2/3)C$, implying $\alpha = -2/3$. Indeed, $1/\alpha = -1.5$ is an eigenvalue of A. We set the target $\delta = 10$. This implies from Eq.(15) that if u_2 satisfies

$$-CA^{-1}B[u_1 \ u_2]^T = -[2/3 \ \ 4/3][u_1 \ u_2]^T = 10, \tag{22}$$

for any measured exogenous input u_1, then $\lim_{t\to\infty} y(t) = 10$. This is confirmed in Fig.6(b). In this simulation, initial values $X(0) = [50 \ \ -20]^T$ were chosen arbitrarily. At $t = 20$s the disturbance input u_1 was changed from 8 to -15 with an underdamped second order dynamics. Note that u_2 is shaped using Eq.(22). Moreover, to demonstrate what was mentioned in *Remark 1*, another simulation was run using a different A matrix with the same B and C and the same inputs (u_1 and u_2). In this case, system was chosen to be

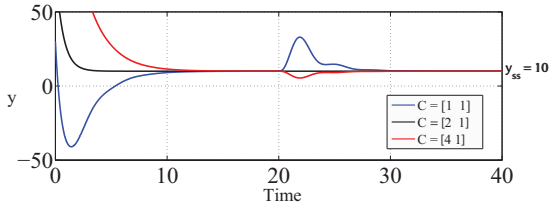

Figure 7. Comparison Of Transient Response For Three Different C Matrices.

$$A = \begin{bmatrix} 2.5 & 1.5 \\ -8 & -4.5 \end{bmatrix} \quad B = \begin{bmatrix} 1 & -2 \\ -3 & 2 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 1 \end{bmatrix} \tag{23}$$

for which the same exact inputs are implemented. In Fig.7 we compare the transient performance of the system described in Eq.(21) with two different choices of C that are not linearly dependent on CA^{-1}. It can be seen from the Fig.7 that with only linearly dependent C, perfect transient response is achievable while for other options, some deviations from the steady state value can be observed.

Complex Conjugate case

So far, we showed the feasibility of transient control for the case where C and CA^{-1} are linearly dependent. This implies that the depenedence is through a negative real eigenvalue; in other words, A must be Hurwitz. However, one could pose the same question for a case where A has complex conjugate yet stable eigenvalues. Now, the question rising from what we have discussed so far is: Considering an LTI-MISO system for which complex conjugate eigenvalues are present; Assuming all the aforementioned assumptions hold through, can we design $u = q(w)$ such that $\lim_{t\to\infty} y(t) = \delta$ where δ is a desired/target value. To answer, we represent the following theorem.

THEOREM 2: *Consider the LTI system as described in Eq.(13) for which all the assumptions mentioned in Theorem 1 are valid with the exception that $y \in \mathbb{R}^{2\times 1}$, $C \in \mathbb{R}^{2\times n}$, and y is such that*

$$CA^{-1} = NC \tag{24}$$

where $N \in \mathbb{R}^{2\times 2}$ is of the form of a real jordan block [13] as follows:

$$N = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \tag{25}$$

with $\alpha < 0, \beta > 0$. Then it is possible to completely eliminate the effect of w on any desired output, say y_1, and ensure $\lim_{t\to\infty} y_1(t) = \delta$ for any transients in w if the L_0 norm of u is greater than or equal to 1.

Proof: This is a scenario where C is related to CA^{-1} by a jordan block, N , which implies the existence of complex conjugate eigenvalues in A.

Now, back to Eq.(13) and (14), for a case with two inputs and two outputs we have

$$\dot{y} = CAX + CB(u+w) \Rightarrow \begin{bmatrix} \dot{y_1} \\ \dot{y_2} \end{bmatrix} = CAX + CB(u+w) \tag{26}$$

Analogous to Eq.(17),

$$C = NCA \Rightarrow CA = N^{-1}C \tag{27}$$

Subsequently, plugging Eq.(27) into Eq.(26),

$$\begin{bmatrix} \dot{y_1} \\ \dot{y_2} \end{bmatrix} = N^{-1} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + CB(u+w) \tag{28}$$

On the other hand, similar to Eq.(14), we have

$$\begin{bmatrix} y_{1_{ss}} \\ y_{2_{ss}} \end{bmatrix} = -NCB(u+w) \tag{29}$$

Hence, by choosing u to satisfy

$$\delta = M_1(u+w) \tag{30}$$

where

$$\begin{bmatrix} M_1 \\ \cdots \\ M_2 \end{bmatrix} = -NCB \tag{31}$$

and noting that $M_1 \in \mathbb{R}^{1\times n}$, we ensure that $\lim_{t\to\infty} y_1(t) = \delta$. This completes the proof. $\diamond\diamond\diamond$

PROBLEM FORMULATION IN FREQUENCY DOMAIN

Since we are interested in multi-variable systems, it prompts us to formulate the problem from singular value/singular vector perspective. Specifically, it is clear that if the input vector is directed along any of the singular vectors corresponding to $\underline{\sigma} = 0$, then y would be completely isolated from the exogenous inputs. Thus, the transient control problem can be posed as follows: *Are there conditions under which it is possible to determine a singular vector corresponding to a zero singular value of the above-mentioned LTI-MISO system without the knowledge of the A matrix?*

Note: A MISO transfer function $C(sI-A)^{-1}B$ will have one or more zero singular values, i.e. $\underline{\sigma} = 0$.

THEOREM 3: *Consider the LTI system described by Eq.(13) and (14)for which all the assumptions outlined in the Theorem 1 are satisfied. If y is such that CA^{-1} and C are linearly dependent, then the direction perpendicular to CB is the direction of a singular vector corresponding to $\underline{\sigma} = 0$. Hence, if the input $(u+w)$ is directed in this direction, then y will be completely isolated from the effect of disturbance w and ensure $\lim_{t\to\infty} y(t) = 0$ for any transient in w.*

Proof: Taking Laplcae transform of Eq.(13)

$$C(sI-A)X(s) = CB[u(s)+w(s)] \tag{32}$$

Therefore, from Eq.(17) we have,

$$\begin{gathered} (s-\tfrac{1}{\alpha})CX(s) = CB[u(s)+w(s)] \Rightarrow \\ y(s) = CB\tfrac{1}{(s-\frac{1}{\alpha})}[u(s)+w(s)] \end{gathered} \tag{33}$$

From Eq.(33), it can be inferred that if u is designed such that $(u(s)+w(s))$ is perpendicular to CB, then complete disturbance rejection can be achieved. This completes the proof. $\diamond\diamond\diamond$

Corollary: *For the above mentioned system under the same assumptions, any desired steady state value for the output δ can be obtained, i.e. $\lim_{t\to\infty} y(t) = \delta$.*

Proof: With,

$$CB[u(t)+w(t)] = -\delta/\alpha \tag{34}$$

Then referring to Eq.(33) We have

$$y(s) = \frac{1}{(s-1/\alpha)}(-\frac{\delta}{\alpha})(\frac{1}{s}) \Rightarrow \lim_{t\to\infty} y(t) = \lim_{s\to 0} \frac{-\delta/\alpha}{s-1/\alpha} = \delta \tag{35}$$

Remark Theorem 2 is indeed, the extension of the application of "Zero" in a multivariable system. According to the definition [14], if $G(s)$ has a zero at $s = z$, then $G(s)$ loses rank at $s = z$ and there will exist a non-zero vector u_z such that

$$G(z)u_z = 0 \tag{36}$$

Where u_z is called zero direction corresponding to a zero singular value of $G(z)$. Similar concept is applied in the Theorem 2;

However, the approach in Theorem 2 is more broad as it works over all frequencies. In other words, the vector of inputs u corresponding to the zero singular value is shaped such that the output y tends to zero not in a certain frequency but over all range of frequencies.

TRANSIENT CONTROL IN LINEARIZED FUEL CELL MODEL

Here, we will represent the state space equations for the fuel cell system which will consist two inputs and one output. we define the states in terms of different mole fractions as follows:

$$Z_1 = (4\mathcal{X}_{1,a} + \mathcal{X}_{2,a} + \mathcal{X}_{4,a}), \quad Z_2 = (4\mathcal{X}_{1,r} + \mathcal{X}_{2,r} + \mathcal{X}_{4,r}) \tag{37}$$

Therefore, the output equation which results in utilization, in terms of states will be:

$$U = y = 1 - \frac{\dot{N}_o(t)Z_1}{\dot{N}_{in}(t)Z_2} \Rightarrow y = C(Z,t) \tag{38}$$

Using the pre-defined states, the state space equation of the fuel cell system will be as follows:

$$\begin{aligned} \dot{Z}_1 &= -\tfrac{\dot{N}_{in}(t)Z_1}{N_r(t)} + k\tfrac{\dot{N}_o(t)Z_2}{N_r(t)} + \tfrac{u_1}{N_r(t)} \\ \dot{Z}_2 &= \tfrac{\dot{N}_{in}(t)Z_1}{N_a(t)} - \tfrac{\dot{N}_o(t)Z_2}{N_a(t)} - \tfrac{u_2}{N_a(t)} \\ \begin{bmatrix}\dot{Z}_1 \\ \dot{Z}_2\end{bmatrix} &= \underbrace{\begin{bmatrix} -\frac{\dot{N}_{in}(t)}{N_r(t)} & k\frac{\dot{N}_o(t)}{N_r(t)} \\ \frac{\dot{N}_{in}(t)}{N_a(t)} & -\frac{\dot{N}_o(t)}{N_a(t)}\end{bmatrix}}_{A(t)} \begin{bmatrix}Z_1 \\ Z_2\end{bmatrix} + \underbrace{\begin{bmatrix}\frac{1}{N_r(t)} & 0 \\ 0 & -\frac{1}{N_a(t)}\end{bmatrix}}_{B(t)} \begin{bmatrix}u_1 \\ u_2\end{bmatrix} \end{aligned} \tag{39}$$

It can be seen from the above equations that fuel cell system is non-autonomous, time varying with nonlinear output. It is highly demanding to work with such a system. Hence, to simplify, we linearize the state-space equations around the steady state point as follows:

Note: The magnitude of N_a and N_r are in the same order;Thence, for simplicity we neglect their appearance in the equations.

$$\begin{aligned} \delta\dot{Z} &= A\delta Z + \delta u \\ \delta\dot{y} &= \tfrac{\partial C}{\partial Z}|_{ss} A\delta Z + \tfrac{\partial C}{\partial Z}|_{ss}\delta u \end{aligned} \tag{40}$$

Where $\frac{\partial C}{\partial Z}|_{ss}$ can be obtained from Eq.(38),

$$\frac{\partial C}{\partial Z}|_{ss} = \frac{\dot{N}_o}{\dot{N}_{in}Z_1^2}\begin{bmatrix} Z_2 & -Z_1\end{bmatrix} \tag{41}$$

Plugging Eq.(41) into Eq.(40),

$$\frac{\partial C}{\partial Z}|_{ss} A = \frac{\dot{N}_o}{\dot{N}_{in}Z_1^2}\begin{bmatrix} -\dot{N}_{in}(Z_1+Z_2) & \dot{N}_o(Z_1+kZ_2)\end{bmatrix} \tag{42}$$

Now we would like to verify whether Theorem 1 can be applied to the linearized fuel cell model or not. Therefore, based on the Theorem 1, linear dependency condition similar to what was expressed earlier that needs to be satisfied to obtain perfect U response under transient conditions will be:

$$\alpha\frac{\partial C}{\partial Z}|_{ss} = \frac{\partial C}{\partial Z}|_{ss} A \Rightarrow -\alpha\frac{Z_2}{Z_1} = \frac{-\dot{N}_{in}(Z_1+Z_2)}{\dot{N}_o(Z_1+kZ_2)} \tag{43}$$

In order to be able to use theorem 1 to justify fuel cell's transient behavior, we decompose $\frac{\partial c}{\partial Z}|_{ss}A$ such that there is a linear dependent part and and a perturbation part. To do that, we first build the linear dependent part from Eq.(43),

$$\begin{aligned} -\tfrac{Z_2}{Z_1} = \tfrac{-\dot{N}_{in}(Z_1+Z_2)}{E} \Rightarrow \\ E = \dot{N}_{in}\tfrac{Z_1}{Z_2}(Z_1+Z_2) = \tfrac{\dot{N}_o}{1-y_{ss}}(Z_1+Z_2) \end{aligned} \tag{44}$$

Now we add and subtract the term $E\frac{\dot{N}_o}{N_{in}Z_1^2}\delta Z_2$ to the Eq.(40),

$$\delta\dot{y} = \frac{\partial C}{\partial Z}|_{ss} A\delta Z + \frac{\partial C}{\partial Z}|_{ss}\delta u + E\frac{\dot{N}_o}{N_{in}Z_1^2}\delta Z_2 - E\frac{\dot{N}_o}{N_{in}Z_1^2}\delta Z_2 \tag{45}$$

Substituting the second component of $\frac{\partial C}{\partial Z}|_{ss}A\delta Z$ by $E\frac{\dot{N}_o}{N_{in}Z_1^2}\delta Z_2$ and subtracting the replaced component, we can use Eq.(43) and write:

$$\delta\dot{y} = \frac{\partial C}{\partial Z}|_{ss}\delta Z - E\frac{\dot{N}_o}{N_{in}Z_1^2}\delta Z_2 + \frac{\dot{N}_o}{N_{in}Z_1^2}(kZ_2+Z_1)\delta Z_2 + \frac{\partial C}{\partial Z}|_{ss}\delta u \tag{46}$$

Similar to the Theorem 1, we factor the output out of the Eq.(46),

$$\delta\dot{y} = -\gamma\delta y + \underbrace{\frac{\dot{N}_o}{N_{in}Z_1^2}\begin{bmatrix}\frac{Z_2+Z_1}{1-y_{ss}} & -(kZ_2+Z_1)\end{bmatrix}}_{P}\delta Z_2 + \frac{\partial c}{\partial Z}|_{ss}\delta u \tag{47}$$

Eventually, considering the inputs,

$$\delta\dot{y} = \underbrace{-\gamma\delta y}_{Homogeneous} + \underbrace{P\delta Z_2 + \frac{\dot{N}_o}{N_{in}Z_1^2}(Z_2\delta u_1 - Z_1\delta u_2)}_{Particular} \tag{48}$$

As γ is a positive number, similar to Theorem 1, the homogeneous part in Eq.(46) will attenuate as time tends toward infinity. Therefore, the question is whether the particular answer of this differential equation using control input can go toward zero or not?

Due to the existence of the perturbation term (P), the condition of Theorem 1 can not be met. However, the reason that an autonomous time-varying system such as fuel cell can have a very acceptable transient response using linear equations is that the

magnitude and significance of the perturbation term in comparison with the control input term is negligible. Therefore, with designing fuel flow rate based on the current draw, a reasonably good transient response in U can be achieved.

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CONCLUSION

In conclusion, a method for controlling transient response of the output of a system for which sensing or accurate estimation was assumed not to be available and comprehensive information regarding plant was not reachable was proposed. First, the area from where this problem was raised was described. Then it was shown through both analysis and simulation that under certain conditions, perfect disturbance attenuation is possible. The method was then verified in frequency domain as well. Thereafter, the conditions under which transient control was possible were relaxed and a more general Theorem was expressed and proved. At the end, the approach were related to fuel cell and similarity and differences were discussed. It was shown that even though fuel cell system is nonlinear, and time-varying, a linear system approach can still be applied yielding a reasonable transient response for the output.

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