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Exponential stabilization of the rolling sphere☆

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Abstract

In this paper we present a non-smooth controller for exponential stabilization of the sphere. This has remained an open problem despite significant progress in nonholonomic systems. Our control design is based on inputs in a rotating coordinate frame that individually produce primitive motions of the sphere along straight lines and circular arcs. The rotating coordinate frame is chosen in concert with Euler angle description of orientation and placement of the desired configuration at the singularity of the representation. In our paper, we separately establish global stability of the desired configuration and exponential convergence. Our theoretical claims are validated through numerical simulations.

Keywords: Rolling sphere; Nonholonomic systems; Exponential stabilization; Euler angles

1. Introduction

For nonholonomic systems, standard nonlinear control methods do not lend themselves well for stabilization to an equilibrium state. This follows from a well-known theorem (Brockett, Millman, & Sussmann, 1982) which establishes that there exists no smooth static state feedback that guarantees asymptotic stability of the equilibrium. To circumvent this problem, researchers have developed strategies that are classified under smooth time-varying stabilization, e.g. (Coron, 1992; Pomet, 1992), piecewise-smooth time-invariant stabilization, e.g. (Aicardi, Casalino, Bicchi, & Balestrino, 1995), and hybrid stabilization, e.g. (Bloch, Reyhanoglu, & McClamroch, 1992). The paper by Coron (1992), in particular, provides general existence results but these results cannot be used to derive explicit control laws. An extensive literature survey of nonholonomic systems can be found in Kolmanovsky and McClamroch (1995).

An important class of nonholonomic systems is the class of two-input nilpotentizable systems that can be transformed into "chained form" (Kolmanovsky & McClamroch, 1995). The chained-form, by its very construction, lends itself well to the development of control systems and researchers have largely focused their efforts on such systems. Those systems that cannot be converted to chained-form require custom design of stabilization strategies—a good example is the work on planar space robots (Mukherjee & Kamon, 1999). The kinematic model of the rolling sphere also cannot be converted to chained-form (Marigo & Bicchi, 2000), and requires custom design of stabilization strategies.

The kinematic model of the sphere is a controllable system (Marigo & Bicchi, 2000), and although many researchers (see Bloch, Krishnaprasad, Marsden, & Murray, 1996; Mukherjee, Minor, & Pukrushpan, 2002, and references therein) have addressed the planning problem, few have investigated the stabilization problem. Date, Sampei, Yamada, Ishikawa, and Koga (1999) designed a controller for converging all states of the ball–plate system to the equilibrium but stability of the equilibrium was not adequately investigated. Oriolo and Vendittelli (2001) showed that the equilibrium of the sphere kinematic model, modelled by five states and two inputs, can be stabilized through iterative

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application of an appropriate open-loop control law. They proposed converging three states of the sphere first and closed trajectories of these states next to steer the two other states closer to their desired coordinates. This idea was proposed earlier by Bloch et al. (1992). The main drawback of the algorithm is that both parts of the controller have to be repeated in the presence of perturbations.

The control problem of the rolling sphere is important for many applications such as attitude stabilization of space multi-body systems, e.g. (Rui, Kolmanovsky, & McClamroch, 2000), mobile robot control (see Mukherjee et al. (2002) and references therein) and robotic manipulation, e.g. (Montana, 1995), and to this end we propose a non-smooth strategy for exponential stabilization. In Section 2, we present the kinematic model (Mukherjee et al., 2002) and in Section 3, we design an algorithm for partial reconfiguration. The complete reconfiguration problem is addressed in Sections 4–6, and stability is established in Section 7. Simulation results are provided in Section 8 and concluding remarks in Section 9. Due to space limitations we have eliminated many details. The proofs which are not given here can be found in Das (2002).

2. Background

2.1. Kinematic model

We denote the Cartesian coordinates of the sphere center by $Q \equiv (x, y)$ and the sphere orientation by z-y-z Euler angles (α, θ, ϕ), as shown in Fig. 1(a). As per the z-y-z Euler angle sequence (Greenwood, 1988), the inertially fixed xyz frame is first rotated about the z-axis by α, $-\pi \leqslant \alpha \leqslant \pi$, to obtain frame $x_1y_1z_1$. This frame is then rotated about y_1 by θ, $0 \leqslant \theta \leqslant \pi$, to obtain frame $x_2y_2z_2$. Finally, the $x_2y_2z_2$ frame is rotated about z_2 by ϕ, $-\pi \leqslant \phi \leqslant \pi$, to obtain frame $x_3y_3z_3$, which is fixed to the sphere. The intersection of the sphere surface with the z_3 and x_3 axes locates points P and R.

The reorientation of the sphere refers to the task of bringing P to the vertically upright position, and R, which then lies on the diametrical circle in the xy plane, to lie on the positive x axis. Indeed, this results in $x_3y_3z_3$, the body-fixed axes, to coincide with the inertially fixed axes xyz. This can be achieved with $\theta = 0$, and $\alpha + \phi = 0$, as shown in Fig. 1(b). Therefore, the sphere can be completely reconfigured by satisfying

$$x = 0, \quad y = 0, \quad \theta = 0, \quad \alpha + \phi = 0. \tag{1}$$

We define the new variable β,

$$\beta = \alpha + \phi. \tag{2}$$

Assuming a sphere of unity radius and denoting its angular velocities about the x_1, y_1, z_1 axes as ω_x^1, ω_y^1, ω_z^1, respectively, the state equations for $\omega_z^1 = 0$[1], can be now written as follows:

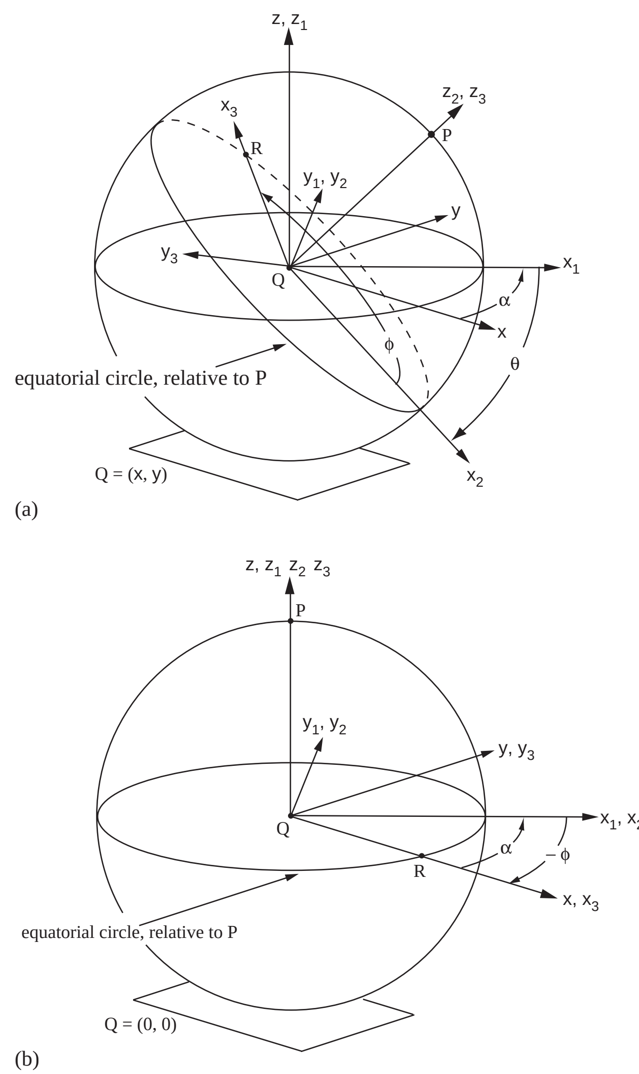


Fig. 1. (a) An arbitrary configuration and, (b) desired configuration of the sphere.

$$\dot{x} = \omega_y^1 \cos\alpha + \omega_x^1 \sin\alpha, \tag{3a}$$

$$\dot{y} = \omega_y^1 \sin\alpha - \omega_x^1 \cos\alpha, \tag{3b}$$

$$\dot{\theta} = \omega_y^1, \tag{3c}$$

$$\dot{\alpha} = -\omega_x^1 \cot\theta, \tag{3d}$$

$$\dot{\beta} = \omega_x^1 \tan(\theta/2). \tag{3e}$$

The above kinematic model has a singularity at $\theta = 0$ and we avoid it by imposing the constraint, $\omega_x^1 = 0$ if $\theta = 0$, on our

[1] The assumption $\omega_z^1 = 0$ is made in conformity with the physical constraint that the sphere cannot spin about the vertical axis.

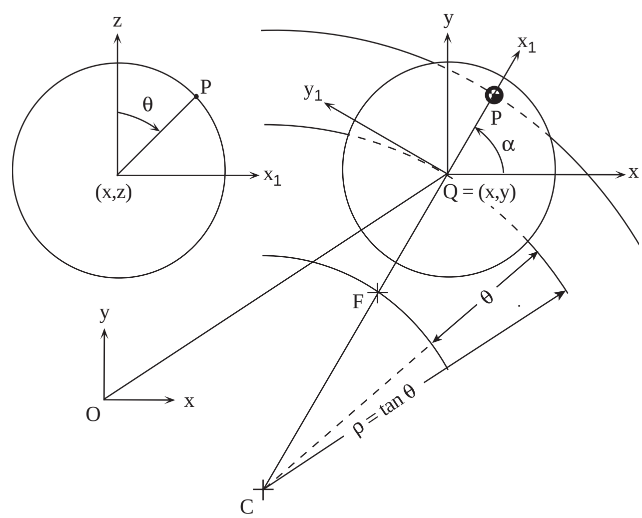


Fig. 2. Motion of the sphere under control actions (A) and (B).

control design. We will achieve complete reconfiguration by satisfying

$$x=0, \quad y=0, \quad \theta=0, \quad \beta=0 \tag{4}$$

independent of the value of α (Mukherjee et al., 2002).

2.2. *Effect of individual control actions*

Consider the motion of the sphere for the individual control actions ω_x^1 and ω_y^1

(A) $\omega_y^1 \neq 0, \quad \omega_x^1 = 0,$
(B) $\omega_x^1 \neq 0, \quad \omega_y^1 = 0, \quad 0<\theta<\pi/2$

as shown in Fig. 2. Under action (A), the sphere moves along line CF as θ changes. Let F be the point on this line where the sphere would have $\theta=0$. Since the sphere rolls without slipping, this point remains invariant under action (A). For action (B), the instantaneous radius of the path traced by the sphere on the xy plane is computed using Eq. (3) as

$$\rho = \left| \frac{(\dot{x}^2+\dot{y}^2)^{3/2}}{\dot{x}\,\ddot{y}-\dot{y}\,\ddot{x}} \right| = \tan\theta = \text{constant}. \tag{5}$$

This implies that the contact point of the sphere moves along a circular path; the center of this circle is located at C in Fig. 2. Along with the contact point, points P and F also move along circular paths; the center of these paths lie on the vertical axis that passes through C. The point C remains fixed under action (B), but under action (A) moves away from F, as θ increases, and converges to F, as θ converges to zero. The variables α, ϕ, and β in Eqs. (2) and (3) change during action (B) but remain constant during action (A). During action (B), the change in β is given by

$$\Delta\beta = \Delta\alpha + \Delta\phi = \Delta\alpha(1-\sec\theta). \tag{6}$$

3. Partial reconfiguration

Our objective for partial reconfiguration is

$$x=0, \quad y=0, \quad \theta=0. \tag{7}$$

We develop our algorithm under the assumption $0<\theta\leqslant(\pi/2-\varepsilon)$ at the initial time, where ε is a small positive number whose choice will be discussed later. This restriction on the initial value of θ will be removed when we develop the complete algorithm for reconfiguration in Section 6. Now, consider the sphere, shown in Fig. 2, described partially by the variables x, y, and θ. The Cartesian coordinates of C, namely, C_x, C_y, are related to those of $Q\equiv(x,y)$, as follows:

$$x = C_x + \tan\theta\,\cos\alpha, \quad y = C_y + \tan\theta\,\sin\alpha. \tag{8}$$

Also, CO and CF are given by the relations

$$CO = \left(C_x^2 + C_y^2\right)^{1/2}, \quad CF = \tan\theta - \theta, \tag{9}$$

where $(\tan\theta-\theta)$ is a monotonically increasing function of θ and equal to zero only when $\theta=0$. It readily follows from Eqs. (8) and (9) that

$$(CF, CO) \equiv (0,0) \Longleftrightarrow (x,y,\theta)\equiv(0,0,0). \tag{10}$$

The basis of our algorithm for partial reconfiguration lies in the theorem presented next.

Theorem 1 (Dual-point theorem). *Let C and F be two points in the xy plane with origin at O. Suppose $\psi=\angle OCF$ is acute, and let $(CF/CO)=n$. If ψ satisfies*

$$0\leqslant\psi<\cos^{-1}(1/n) \quad \text{for } n\in(1,\infty), \tag{11a}$$

$$0\leqslant\psi<\cos^{-1}(n) \quad \text{for } n\in(0,1), \tag{11b}$$

then there exists a point C' on the extended line CF, as shown in Figs. 3(a) and (b), such that for $\psi'=\angle OC'F$, $0\leqslant\psi'\leqslant\pi$,

$$\begin{aligned} &(C'F/C'O)=n \quad \cdots \text{ (i)},\\ &0<(C'O/CO)<1 \quad \cdots \text{ (ii)},\\ &\psi'>\psi \quad \cdots \text{ (iii)}. \end{aligned} \tag{12}$$

Using geometry, it can be shown

$$\tan\psi' = \frac{-\mid 1-n^2\mid \sin\psi}{(1+n^2)\cos\psi-2n}, \quad n\in(0,1)\cup(1,\infty). \tag{13}$$

It can be verified from Eq. (13) that

$$\begin{aligned} &0\leqslant\psi<\cos^{-1}\left(\frac{1}{n}\right) \Rightarrow \cos^{-1}\left(\frac{1}{n}\right)<\psi'\leqslant\pi \\ &\qquad \text{for} \quad n\in(1,\infty),\\ &0\leqslant\psi<\cos^{-1}(n) \Rightarrow \cos^{-1}(n)<\psi'\leqslant\pi \\ &\qquad \text{for} \quad n\in(0,1). \end{aligned} \tag{14}$$

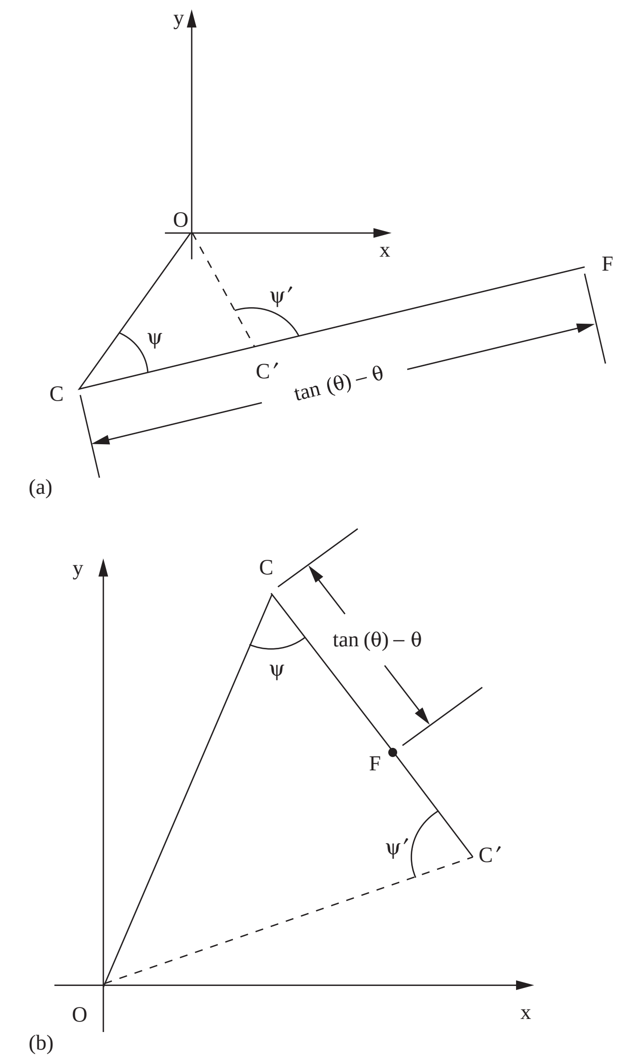


Fig. 3. C–C' pair in the Dual-Point theorem for (a) $n \in (1, \infty)$ and (b) $n \in (0, 1)$.

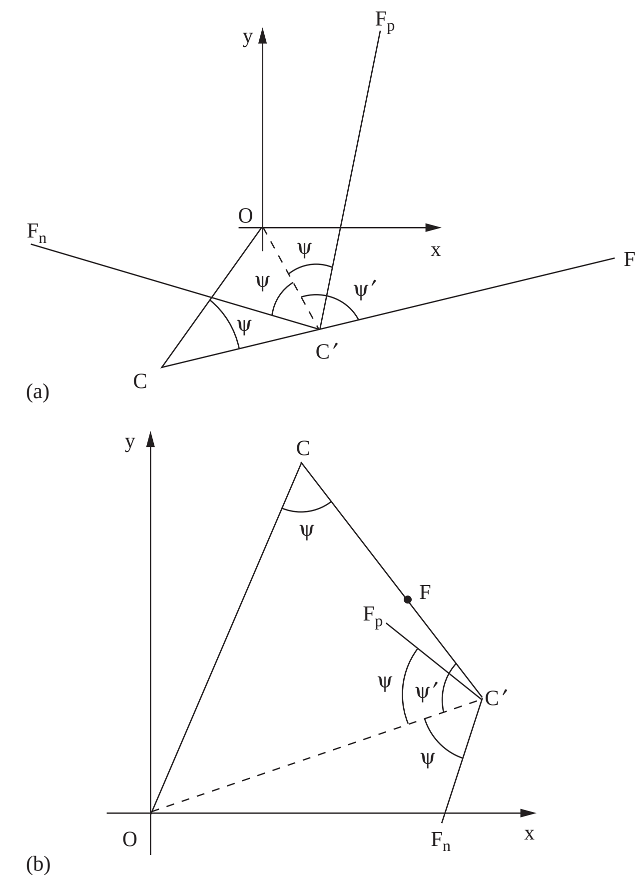


Fig. 4. RS and DPT maneuvers for the two cases (a) $n \in (1, \infty)$ and (b) $n \in (0, 1)$.

The following inequality, stated without proof, will be later useful in our analysis

$$\psi + \psi' < \pi, \quad n \in (0, 1) \cup (1, \infty). \tag{15}$$

Consider the two cases $n \in (1, \infty)$ and $n \in (0, 1)$ shown in Fig. 4. Suppose $\psi = \angle OCF$ satisfies Eq. (11) and C' satisfies Eq. (12). We now define three specific maneuvers of the sphere.

Definition 1 (DPT maneuver). In reference to Figs. 4(a) and (b), we define a "Dual-Point Tuck" (DPT) Maneuver as control action (A) that moves the sphere such that point C moves to C'.

From Theorem 1, we know that a DPT maneuver results in $\psi' > \psi$. For both cases $n \in (1, \infty)$ and $n \in (0, 1)$, ψ' can therefore be restored to the value ψ in one of two ways as shown in Fig. 4. This motivates the next definition.

Definition 2 (RS maneuver). Following a DPT maneuver, a control action (B) that moves the sphere to restore ψ' to ψ is defined as a "Restoring-Sweep" (RS) maneuver.

Since the initial configuration may not satisfy $\angle OCF = \psi'$, we define one additional maneuver.

Definition 3 (PS maneuver). A control action (B) that moves the sphere at the initial time to bring $\angle OCF$ to ψ' is defined as a "Preliminary-Sweep" (PS) maneuver.

We now present the "Sweep-Tuck" algorithm:

Theorem 2 (Sweep-Tuck algorithm). *Consider a sphere whose partial configuration* (x, y, θ) *is defined by the points C and F. Suppose at the initial time,* $0 < \theta \leqslant (\pi/2 - \varepsilon)$ *and* $(CF/CO) = n \in (0, 1) \cup (1, \infty)$. *If* ψ *is chosen in accordance with Eq.* (11), *partial reconfiguration in the sense of Eq.* (7) *can be achieved through a PS maneuver followed by repeated RS–DPT maneuvers.*

Proof. The application of a PS maneuver results in $\angle OCF = \psi'$. This sets the stage for repeated application of RS–DPT maneuvers. The application of an RS maneuver does not alter the values of CF and CO but sweeps F about C to bring $\angle OCF$ to the value ψ, chosen in accordance with Eq. (11). At the end of the RS maneuver, the new point, F_p or F_n, is simply renamed F. Using Theorem 1 we can show that a subsequent DPT maneuver results in

$$0 < C'O/CO = C'F/CF = r < 1,$$
$$r \triangleq \left| 1 - \frac{2(n\cos\psi - 1)}{(n^2-1)} \right| \tag{16}$$

and change of $\angle OCF = \psi$ to $\angle OC'F = \psi'$, $\psi' > \psi$, as shown in Figs. 3 and 4. By renaming C' as C, we can again execute the RS–DPT pair. Each pair reduces CF and CO in geometric progression and if $\{CF_1, CF_2, \ldots, CF_k, \ldots\}$, and $\{CO_1, CO_2, \ldots, CO_k, \ldots\}$ denote the sequence of values of CF and CO at the end of the RS–DPT pairs, respectively, then

$$CF_k = r^{k-1}CF_1, \quad CO_k = r^{k-1}CO_1$$

and $CF_k, CO_k \to 0$ as $k \to \infty$, since $r \in (0,1)$. Using Eq. (10) we claim partial reconfiguration. □

Corollary 1. *The sequence of values assumed by θ at the end of every DPT maneuver of the sweep-tuck algorithm decreases monotonically and converges to zero.*

4. Complete reconfiguration: convergence studies for $n \in (1, \infty)$

4.1. Problem statement

In this section, we extend the sweep-tuck algorithm to additionally converge $\beta \to 0$ for $n \in (1, \infty)$. As in Section 3, the algorithm will be developed under the restriction $0 < \theta \leqslant (\pi/2 - \varepsilon)$. In the next section, we will address the complete reconfiguration problem for $n \in (0, 1)$ and in Section 6, we will address special cases that require some initial maneuvers, such as the case where $\theta > (\pi/2 - \varepsilon)$, and cases where $n = 0$, 1, and ∞.

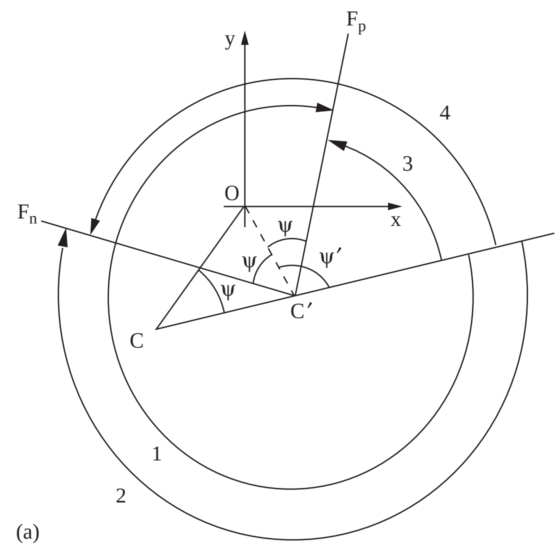
4.2. Analysis of quadruple sweep options

To investigate the change in β for four RS options, we define the following:

Definition 4 (P_ψ configuration). The pair $\{C, F\}$, with acute $\angle OCF = \psi$, defines a P_ψ configuration if $\overrightarrow{CF} \times \overrightarrow{CO} > 0$.

Definition 5 (N_ψ configuration). The pair $\{C, F\}$, with acute $\angle OCF = \psi$, defines an N_ψ configuration if $\overrightarrow{CF} \times \overrightarrow{CO} < 0$.

In Fig. 5(a), $\{C, F\}$ and $\{C', F_p\}$ are P_ψ configurations, $\{C', F\}$ is a $P_{\psi'}$ configuration and $\{C', F_n\}$ is an N_ψ configuration. We now investigate the change in β for the four RS maneuvers that are possible starting from $P_{\psi'}$: $\{C', F\}$ in Fig. 5(a). These maneuvers, marked 1, 2, 3, and 4, respectively, correspond to



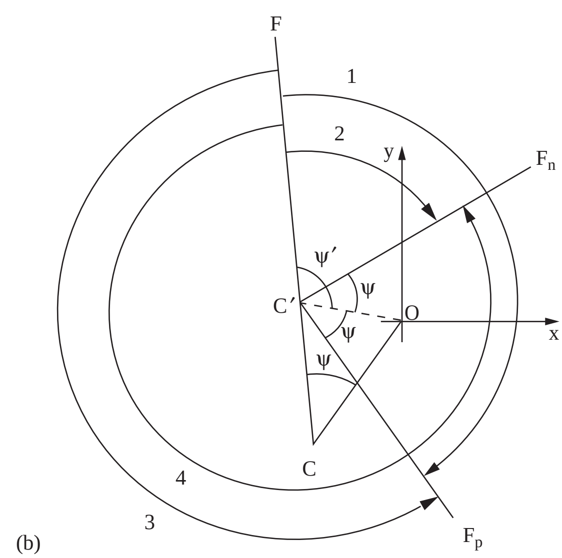


Fig. 5. Quadruple options for RS maneuvers starting from (a) $P_{\psi'}$ and (b) $N_{\psi'}$ configuration.

1. a cw sweep ending at P_ψ: $\{C', F_p\}$,
2. a cw sweep ending at N_ψ: $\{C', F_n\}$,
3. a ccw sweep ending at P_ψ: $\{C', F_p\}$, and
4. a ccw sweep ending at N_ψ: $\{C', F_n\}$.

It can be verified from Fig. 2 that the angle of sweep during an RS maneuver is equal to $\Delta\alpha$. For the above maneuvers, $\Delta\beta$ can therefore be computed using Eq. (6); the results are summarized in Tables 1 and 2 for $P_{\psi'}$: $\{C', F\}$ and $N_{\psi'}$: $\{C', F\}$ in Figs. 5(a) and (b), respectively.

It was established in Theorem 1 that $\psi' > \psi$ for $n \in (1, \infty) \cup (0, 1)$. Furthermore, we know from Eq. (15) that $\psi + \psi' < 2\pi$.

Table 1
Analyses of sweep options from $P_{\psi'}$ configuration

Start configuration: $P_{\psi'}$			
Direction	End configuration	Sweep angle	Change in variable β
cw	P_ψ	$\Delta\alpha_1 = -(2\pi - \psi' + \psi)$	$\Delta\beta_1 = -(2\pi - \psi' + \psi)(1 - \sec\theta)$
cw	N_ψ	$\Delta\alpha_2 = -(2\pi - \psi' - \psi)$	$\Delta\beta_2 = -(2\pi - \psi' - \psi)(1 - \sec\theta)$
ccw	P_ψ	$\Delta\alpha_3 = \psi' - \psi$	$\Delta\beta_3 = (\psi' - \psi)(1 - \sec\theta)$
ccw	N_ψ	$\Delta\alpha_4 = \psi' + \psi$	$\Delta\beta_4 = (\psi' + \psi)(1 - \sec\theta)$

Table 2
Analyses of sweep options from $N_{\psi'}$ configuration

Start configuration: $N_{\psi'}$			
Direction	End configuration	Sweep angle	Change in variable β
cw	P_ψ	$\Delta\alpha_1 = -(\psi' + \psi)$	$\Delta\beta_1 = -(\psi' + \psi)(1 - \sec\theta)$
cw	N_ψ	$\Delta\alpha_2 = -(\psi' - \psi)$	$\Delta\beta_2 = -(\psi' - \psi)(1 - \sec\theta)$
ccw	P_ψ	$\Delta\alpha_3 = 2\pi - \psi' - \psi$	$\Delta\beta_3 = (2\pi - \psi' - \psi)(1 - \sec\theta)$
ccw	N_ψ	$\Delta\alpha_4 = 2\pi - \psi' + \psi$	$\Delta\beta_4 = (2\pi - \psi' + \psi)(1 - \sec\theta)$

Therefore, the following relations hold between the four possible sweep angles in Table 1

$$-(2\pi - \psi' + \psi) \leqslant -(2\pi - \psi' - \psi) \leqslant 0 \leqslant (\psi' - \psi) \leqslant (\psi' + \psi). \tag{17a}$$

Similarly, the sweep angles in Table 2 satisfy

$$-(\psi' + \psi) \leqslant -(\psi' - \psi) \leqslant 0 \leqslant (2\pi - \psi' - \psi) \leqslant (2\pi - \psi' + \psi). \tag{17b}$$

4.3. Compensating and restoring sweep (CRS)

Let $(x_0, y_0, \theta_0, \alpha_0, \beta_0)$ be the initial configuration of the sphere. A PS maneuver is first invoked to set $\angle OCF = \psi'$. Let $(x_1, y_1, \theta_1, \alpha_1, \beta_1)$ be the configuration variables at the end of this maneuver. If we denote all configuration variables prior to the kth RS–DPT pair using subscript k, $(x_1, y_1, \theta_1, \alpha_1, \beta_1)$ denotes the configuration variables prior to application of the first RS–DPT pair. The change in β during the kth RS–DPT pair is

$$\beta_{k+1} = \beta_k + \Delta\beta, \tag{18}$$

where $\Delta\beta$ takes the values in Tables 1 and 2 for start configurations $P_{\psi'}$ and $N_{\psi'}$, respectively. We now define the compensating and restoring sweep maneuver.

Definition 6 (CRS maneuver). Among the four choices for an RS maneuver in a sweep-tuck sequence, the compensating and restoring sweep (CRS) maneuver is the one that minimizes the absolute value of β, i.e.,

$$|\beta_{k+1}| = \min_{\Delta\beta \in S} |\beta_k + \Delta\beta|, \tag{19}$$

where $S = \{\Delta\beta_1, \Delta\beta_2, \Delta\beta_3, \Delta\beta_4\}$, and $\Delta\beta_1$, $\Delta\beta_2$, $\Delta\beta_3$, and $\Delta\beta_4$ are the entries in Tables 1 or 2 depending on whether the configuration variables define a $P_{\psi'}$ or $N_{\psi'}$ configuration, respectively.

We now investigate the effect of a CRS maneuver for a $P_{\psi'}$ start configuration. The range of the set of entries of S in Eq. (19), taken from Table 1, is

$$(\psi + \psi')(1 - \sec\theta_k) \leqslant S \leqslant -(2\pi - \psi' + \psi)(1 - \sec\theta_k). \tag{20}$$

Suppose β_k lies in the range that is a mirror image of the range of S in Eq. (20). This implies

$$(2\pi - \psi' + \psi)(1 - \sec\theta_k) \leqslant \beta_k \leqslant -(\psi + \psi')(1 - \sec\theta_k). \tag{21}$$

The range of β_{k+1} can now be obtained using Eqs. (19), (20) and (21). This range reveals that $\beta_{k+1} = 0$ when $\beta_k = -\Delta\beta_i$, $i = 1, 2, 3, 4$. For other values of β_k in the range given by Eq. (21), β_{k+1} varies piecewise linearly and $|\beta_{k+1}|$ reaches a local maxima of $-\psi(1 - \sec\theta_k)$ when $\beta_k = (-\Delta\beta_1 - \Delta\beta_2)/2$ and $\beta_k = (-\Delta\beta_3 - \Delta\beta_4)/2$, and the global maxima of $(\psi - \pi)(1 - \sec\theta_k)$ when $\beta_k = (-\Delta\beta_2 - \Delta\beta_3)/2$. Since the global maxima of $|\beta_{k+1}|$ is $(\psi - \pi)(1 - \sec\theta_k)$, we can reduce the conservatism of the range of β_k in Eq. (21) by expanding it by $(\psi - \pi)(1 - \sec\theta_k)$ on both sides. Mathematically, the expanded range can be expressed as follows:

$$(3\pi - \psi')(1 - \sec\theta_k) \leqslant \beta_k \leqslant -(\pi + \psi')(1 - \sec\theta_k), \tag{22}$$

and it guarantees

$$|\beta_{k+1}| \leqslant (\psi - \pi)(1 - \sec\theta_k). \tag{23}$$

If the CRS maneuver has a $N_{\psi'}$ start configuration, the extended range of β_k is found to be

$$(\pi + \psi')(1 - \sec\theta_k) \leqslant \beta_k \leqslant -(3\pi - \psi')(1 - \sec\theta_k). \tag{24}$$

The next lemma summarizes these results:

Lemma 1. *Consider a sequence of CRS–DPT maneuvers. If the configuration variables prior to the kth* ($k \geqslant 1$) *CRS–DPT pair define a* $P_{\psi'}$ *configuration and satisfy Eq.* (22), *or define a* $N_{\psi'}$ *configuration and satisfy Eq.* (24), β_{k+1} *will be bounded according to Eq.* (23).

4.4. Inequality condition for convergence

The bounds on β_{k+1} in Eq. (23) are valid for the expanded range of β_k in Eqs. (22) and (24) for $P_{\psi'}$ and $N_{\psi'}$ start configurations, respectively. For sub-intervals within the range, if we define

$$\mu_k = (\psi - \pi)\,(1 - \sec\theta_k), \quad v_k = -\psi\,(1 - \sec\theta_k) \tag{25}$$

we have the following Lemma:

Lemma 2. *Consider a sequence of CRS–DPT maneuvers. If the configuration variables prior to the kth* ($k \geqslant 1$) *CRS–DPT pair define a* $P_{\psi'}$ *configuration and satisfy Eq.* (22), *or define a* $N_{\psi'}$ *configuration and satisfy Eq.* (24), β_{k+1} *will be bounded according to the relation*

$$-\mu_k \leqslant \beta_{k+1} \leqslant v_k \tag{26a}$$

if the CRS maneuver ends in a P_{ψ} *configuration, and according to the relation*

$$-v_k \leqslant \beta_{k+1} \leqslant \mu_k \tag{26b}$$

if the CRS maneuver ends in a N_{ψ} *configuration.*

Using Lemma 2 we arrive at the first reconfiguration theorem stated next.

Theorem 3 (First reconfiguration theorem). *Consider the sweep-tuck algorithm for* $n \in (1, \infty)$ *and* ψ *satisfying Eq.* (11). *Suppose* $0 < \theta_0 \leqslant (\pi/2 - \varepsilon)$ *as required by the algorithm. Let k,* $k \geqslant 1$, *be any integer for which the configuration variables* $(x_k, y_k, \theta_k, \alpha_k, \beta_k)$ *define a* $P_{\psi'}$ *configuration and satisfy Eq.* (22) *or define a* $N_{\psi'}$ *configuration and satisfy Eq.* (24). *If for all integer values of* j, $j \geqslant k$, *the jth RS maneuver is a CRS and the inequality*

$$\frac{(1 - \sec\theta_j)}{(1 - \sec\theta_{j+1})} \leqslant \frac{(\pi + \psi')}{(\pi - \psi)} \tag{27}$$

is satisfied, then $(x_j, y_j, \theta_j, \beta_j) \to (0, 0, 0, 0)$ *as* $j \to \infty$ *and the sphere is completely reconfigured.*

4.5. Range of ψ for inequality condition

In this section, we establish that the inequality condition in Eq. (27) is always satisfied for a subset of the range of ψ in Eq. (11a). To this end, we first note from Eq. (14) that ψ and ψ' lie in the ranges $0 \leqslant \psi < \cos^{-1}(1/n)$ and $\cos^{-1}(1/n) \leqslant \psi' < \pi$, respectively. Using Eq. (13) we can readily show that $\psi' = \cos^{-1}(1/n)$ when $\psi = \cos^{-1}(1/n)$. Thus,

$$\lim_{\psi \to \cos^{-1}(1/n)} \frac{(\pi + \psi')}{(\pi - \psi)} = \frac{\pi + \cos^{-1}(1/n)}{\pi - \cos^{-1}(1/n)} > 1. \tag{28}$$

Using the expression for r in Eq. (16) we can show

$$\lim_{\psi \to \cos^{-1}(1/n)} \frac{C'F}{CF} = 1. \tag{29}$$

From Eqs. (9) and (29) we can therefore deduce that for $\psi \to \cos^{-1}(1/n)$,

$$\frac{\tan\theta_{j+1} - \theta_{j+1}}{\tan\theta_j - \theta_j} = 1 \Rightarrow \theta_{j+1} = \theta_j, \qquad \Rightarrow \frac{1 - \sec\theta_{j+1}}{1 - \sec\theta_j} = 1. \tag{30}$$

From Eqs. (28) and (30) we conclude that there exists a Ψ, $0 \leqslant \Psi < \cos^{-1}(1/n)$, such that Eq. (27) is always satisfied for $\Psi \leqslant \psi < \cos^{-1}(1/n)$. To compute Ψ, we use Taylor's series expansion to approximate $(\sec\theta - 1) \approx 1.5(\tan\theta - \theta)/\theta$. Then, using Eqs. (9) and (16) and Corollary 1 we obtain

$$\frac{(1 - \sec\theta_j)}{(1 - \sec\theta_{j+1})} < \frac{(\tan\theta_j - \theta_j)}{(\tan\theta_{j+1} - \theta_{j+1})} = \frac{1}{\left[1 - 2(n\cos\psi - 1)/(n^2 - 1)\right]}. \tag{31}$$

Hence, Eq. (27) is satisfied if

$$\frac{1}{\left[1 - 2(n\cos\psi - 1)/(n^2 - 1)\right]} \leqslant \frac{(\pi + \psi')}{(\pi - \psi)}. \tag{32}$$

The value of Ψ can be computed from Eq. (32). Since θ does not appear in Eq. (32), Ψ can be computed a priori from the value of n alone and the data stored in a look-up table for quick reference. We have provided the value of Ψ in radians for specific values of n in Table 3. We now state a corollary of Theorem 3:

Corollary 2. *Consider the sweep-tuck algorithm for* $n \in (1, \infty)$ *and* $\Psi \leqslant \psi \leqslant \cos^{-1}(1/n)$. *Suppose* $0 < \theta_0 \leqslant (\pi/2 - \varepsilon)$, *as required by the algorithm. Let k,* $k \geqslant 1$, *be any integer for which the configuration variables* $(x_k, y_k, \theta_k, \alpha_k, \beta_k)$ *define a* $P_{\psi'}$ *configuration and satisfy Eq.* (22) *or define an* $N_{\psi'}$ *configuration and satisfy Eq.* (24). *If for all integer values of* j, $j \geqslant k$, *the jth RS maneuver is a CRS maneuver, then* $(x_j, y_j, \theta_j, \beta_j) \to (0, 0, 0, 0)$ *as* $j \to \infty$ *and the sphere is completely reconfigured.*

4.6. PS maneuver and merging the extended regions

We assumed the initial configuration of the sphere to be $(x_0, y_0, \theta_0, \alpha_0, \beta_0)$ in Section 4.3. After the PS maneuver, which sets $\angle OCF = \psi'$, we assumed the configuration

Table 3
Numerical values of Ψ for various $n \in (1, \infty)$

n	1.1	1.2	1.25	1.5	1.75	2.0	2.5	2.75	3.0	3.5	4.0
Ψ	0.371	0.473	0.505	0.580	0.584	0.552	0.411	0.293	0.0	0.0	0.0

variables to be $(x_1, y_1, \theta_1, \alpha_1, \beta_1)$. Here we investigate the change in β, $\Delta\beta = (\beta_1 - \beta_0)$, due to the PS maneuver.

Since the maximum angle of pre-sweep can be 2π, the maximum change in β due to the PS maneuver can be computed from Eq. (6) as follows:

$$\Delta\beta_{\max} = \max(\beta_1 - \beta_0) = \pm 2\pi(1 - \sec\theta_0), \tag{33}$$

where the sign is positive for ccw sweep and negative for cw sweep. The expanded range of β for subscript $k = 0$, for both Eqs. (22) and (24) is

$$\begin{aligned}\mathscr{W} &= \{(\pi + \psi') + (3\pi - \psi')\}(1 - \sec\theta_0)\\ &= 4\pi(1 - \sec\theta_0) \geqslant 2\,|\Delta\beta|.\end{aligned} \tag{34}$$

This implies that the direction of pre-sweep can be chosen suitably such that

$$\begin{aligned}&(3\pi - \psi')(1 - \sec\theta_0) \leqslant \beta_0 \leqslant -(\pi + \psi')(1 - \sec\theta_0)\\ &\quad\Rightarrow (3\pi - \psi')(1 - \sec\theta_1) \leqslant \beta_1 \leqslant -(\pi + \psi')(1 - \sec\theta_1)\end{aligned} \tag{35a}$$

and

$$\begin{aligned}&(\pi + \psi')(1 - \sec\theta_0) \leqslant \beta_0 \leqslant -(3\pi - \psi')(1 - \sec\theta_0)\\ &\quad\Rightarrow (\pi + \psi')(1 - \sec\theta_1) \leqslant \beta_1 \leqslant -(3\pi - \psi')(1 - \sec\theta_1).\end{aligned} \tag{35b}$$

Both Eqs. (35a) and (35b) are based on the fact that θ remains constant during a PS maneuver, i.e. $\theta_1 = \theta_0$. We are now ready to define the "Proper Preliminary-Sweep" (PPS) maneuver.

Definition 7 (PPS maneuver). A PS maneuver that satisfies Eq. (35a) or Eq. (35b) is said to be a PPS maneuver.

Theorem 4 (Second reconfiguration theorem). *Let the initial configuration of the sphere satisfy* $n \in (1, \infty)$, $0 < \theta_0 \leqslant (\pi/2 - \varepsilon)$, *and*

$$|\beta_0| \leqslant -(3\pi - \psi')(1 - \sec\theta_0), \tag{36}$$

the sphere can be completely reconfigured by a PPS maneuver followed by repeated application of CRS–DPT pairs with $\psi \in [\Psi, \cos^{-1}(1/n)]$.

5. Complete reconfiguration: convergence studies for $n \in (0, 1)$

The difference between the two cases $n \in (1, \infty)$ and $n \in (0, 1)$ arises from the fact that $CC' < CF$ for $n \in (1, \infty)$ and $CC' > CF$ for $n \in (0, 1)$. This implies that a DPT maneuver changes a P_ψ configuration into a $P_{\psi'}$ configuration and an N_ψ configuration into an $N_{\psi'}$ configuration for $n \in (1, \infty)$, but for $n \in (0, 1)$ it changes a P_ψ configuration into an $N_{\psi'}$ configuration and an N_ψ configuration into a $P_{\psi'}$ configuration. With this fundamental difference in perspective, the analyses of Section 4 can be repeated for $n \in (0, 1)$. The main results are very similar and are stated below without the proofs.

Theorem 5 (Parallel of Theorem 3). *Consider the sweep-tuck algorithm for* $n \in (0, 1)$ *with* ψ *chosen to satisfy Eq.* (11). *Suppose* $0 < \theta_0 \leqslant (\pi/2 - \varepsilon)$, *as required by the algorithm. Let* k, $k \geqslant 1$, *be any integer for which the configuration variables* $(x_k, y_k, \theta_k, \alpha_k, \beta_k)$ *define a* $P_{\psi'}$ *configuration and satisfy Eq.* (22) *or define a* $N_{\psi'}$ *configuration and satisfy Eq.* (24). *If for all integer values of* j, $j \geqslant k$, *the jth RS maneuver is a CRS maneuver and*

$$\frac{(1 - \sec\theta_j)}{(1 - \sec\theta_{j+1})} \leqslant \frac{(\pi + \psi')}{(\pi - \psi)} \tag{37}$$

is satisfied, then $(x_j, y_j, \theta_j, \beta_j) \to (0, 0, 0, 0)$ *as* $j \to \infty$ *and the sphere is completely reconfigured.*

Similar to the case $n \in (1, \infty)$, we can numerically compute Ψ, $0 \leqslant \Psi < \cos^{-1}(n)$, for the case $n \in (0, 1)$ such that Eq. (37) is satisfied. The computed values of Ψ are provided in Table 4.

Theorem 6 (Parallel of Theorem 4). *Let the initial configuration of the sphere satisfy* $n \in (0, 1)$, $0 < \theta_0 \leqslant (\pi/2 - \varepsilon)$, *and* β_0 *in the range*

$$|\beta_0| \leqslant -(3\pi - \psi')(1 - \sec\theta_0), \tag{38}$$

the sphere can be completely reconfigured by a PPS maneuver followed by repeated application of CRS–DPT pairs with $\psi \in [\Psi, \cos^{-1}(n)]$.

6. Complete algorithm for convergence

6.1. Tuck-out maneuver

We have seen from Theorems 4 and 6 in Sections 4 and 5, respectively, that β_0 must satisfy Eq. (38) for the sweep-tuck algorithm to be applicable. If we define

$$\zeta = \begin{cases}\cos^{-1}(1/n) & \text{for } n \in (1, \infty),\\ \cos^{-1}(n) & \text{for } n \in (0, 1),\end{cases} \tag{39}$$

Table 4
Numerical values of Ψ for various $n \in (0, 1)$

n	0.9	0.8	0.7	0.6	0.5	0.4	0.33	0.3	0.25	0.2	0.1
Ψ	0.387	0.505	0.569	0.588	0.552	0.411	0.0	0.0	0.0	0.0	0.0

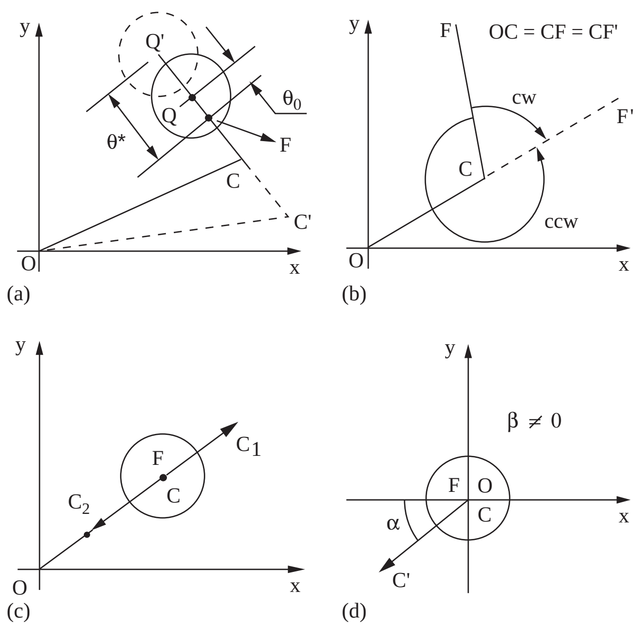


Fig. 6. (a) Tuck-out and initial maneuvers for (b) $n = 1$, (c) $n = 0$, and (d) n is undefined.

the maximum range of β_0 is given by

$$(3\pi - \zeta)\,(1 - \sec\theta_0) \leqslant \beta_0 \leqslant -\,(3\pi - \zeta)\,(1 - \sec\theta_0) \tag{40}$$

since $\zeta < \psi' \leqslant \pi$ from Eq. (14). From the relation between ψ and ψ' in Eq. (13) and their ranges in Eq. (14) we know that $\psi' \to \zeta$ as $\psi \to \zeta$. Thus, if β_0 satisfies the range condition in Eq. (40) for $n \in (0, 1) \cup (1, \infty)$, there will always exist a subrange of ψ given by

$$\bar{\psi} \leqslant \psi \leqslant \zeta \;\Rightarrow\; |\beta_0| = -(3\pi - \bar{\psi}')\,(1 - \sec\theta_0), \tag{41}$$

where $\bar{\psi}'$ relates to $\bar{\psi}$ by Eq. (13), such that Theorems 4 or 6 will be applicable with $\psi \in (\bar{\psi}, \zeta) \cap (\Psi, \zeta)$.

If Eq. (40) is not satisfied, we have $|\beta_0| > -\,(3\pi - \zeta)\,(1 - \sec\theta_0)$. Then define θ^* such that

$$|\beta_0| = -(3\pi - \zeta)\,(1 - \sec\theta^*). \tag{42}$$

This implies $\theta^* > \theta_0$. To satisfy Eq. (40), we use control action (A), whereby β_0 stays constant and θ_0 is increased to the value of θ^*.

Consider the maneuver of the sphere depicted in Fig. 6(a), where points C and Q move to the locations C' and Q', respectively, as θ increases. Clearly, such a maneuver changes the value of $n \triangleq CF/CO$ and therefore Eq. (42) cannot be used to calculate θ^*, since $\zeta = \zeta(n)$. To circumvent the problem, we will have to choose the most conservative (maximum) value of θ^* such that Eq. (42) is satisfied for all values of $n \in (0, 1) \cup (1, \infty)$. This is achieved by using the maximum value of ζ, $\zeta = \pi/2$, in Eq. (42) which now gives us the correct expression for computing θ^* from β_0

$$|\beta_0| = -(3\pi - \pi/2)\,(1 - \sec\theta^*)$$
$$\Rightarrow \theta^* = \cos^{-1}\left(\frac{2.5\pi}{2.5\pi + |\beta_0|}\right). \tag{43}$$

It follows from the above discussion that if the initial value of θ satisfies $\theta_0 \geqslant \theta^*$, where θ^* is defined by Eq. (43), Eq. (40) will be satisfied for any $n \in (0, 1) \cup (1, \infty)$ and Theorems 4 or 6 will be applicable with some value of $\psi \in (\bar{\psi}, \zeta) \cap (\Psi, \zeta)$. If $\theta_0 < \theta^*$, we will increase θ to θ^* based on the maneuver defined next.

Definition 8 (TO maneuver). At the initial time, if $\theta_0 < \theta^*$, where θ^* is defined in Eq. (43), a control action (A) that increases the value of θ to θ^* is defined as a "Tuck-Out" (TO) maneuver.

Remark 1. Since the maximum value of $|\beta_0|$, $|\beta_0|_{\max}$, is equal to π, θ^* has a maximum value of $\theta^*_{\max} = \cos^{-1}[2.5\pi/(2.5\pi + \pi)] = 0.775\ \text{rad} = 44.42^\circ$. This implies that a TO maneuver will not be required if $\theta_0 > 44.42^\circ$.

Remark 2. We have continually assumed that $0 < \theta_0 \leqslant (\pi/2 - \varepsilon)$. Now we discuss the choice of ε. We choose $\varepsilon < (\pi/2 - \theta^*_{\max})$ such that $\theta^* \leqslant \theta^*_{\max} < (\pi/2 - \varepsilon)$. This ensures that Theorems 4 and 6 will be applicable after a TO maneuver. We will however not choose a very small value of ε since it will adversely affect the exponential rate of convergence—as shown in Section 7.

The next theorem combines Theorems 4 and 6 and relaxes the initial range condition on β:

Theorem 7 (Third reconfiguration theorem). *Let the initial configuration of the sphere satisfy* $n \in (0, 1) \cup (1, \infty)$, $0 < \theta_0 \leqslant (\pi/2 - \varepsilon)$, *and*

$$(3\pi - \zeta)\,(1 - \sec\theta_0) \leqslant \beta_0 \leqslant -\,(3\pi - \zeta)\,(1 - \sec\theta_0), \tag{44}$$

then, the sphere can be completely reconfigured by a PPS maneuver followed by repeated application of CRS–DPT pairs with $\psi \in [\max(\bar{\psi}, \Psi), \zeta]$. *If* β_0 *lies outside the range, an additional TO maneuver is required prior to the PPS maneuver.*

6.2. Special cases

Although Theorem 7 removes the range condition on β_0, it still requires the sphere to satisfy $n \in (0,1) \cup (1,\infty)$ and $0 < \theta_0 \leqslant (\pi/2-\varepsilon)$. In this section, we investigate the cases where $n=0$, $n=1$, $n=\infty$, the case where n is undefined, and $\theta_0 > (\pi/2-\varepsilon)$. Note that $\theta_0=0$ is covered by the cases $n=0$ and n is undefined. For each of these special cases, we provide a strategy comprised of atmost two maneuvers that changes the configuration of the sphere to one that satisfies $n \in (0,1)\cup(1,\infty)$ and $0<\theta\leqslant(\pi/2-\varepsilon)$, such that Theorem 7 can be subsequently applied.

$\underline{n=1}$: This occurs when $CF = CO \neq 0$. We investigate two cases, (a) $\theta_0 < \theta^*$, and (b) $\theta_0 \geqslant \theta^*$. For case (a) we propose a TO maneuver to change the value of n. For case (b), we change n by:

1. first using control action (B) to make points O, C, and F collinear, and in that order,
2. then using control action (A) to change n.

Since $\theta_0 \geqslant \theta^*$, Eq. (40) holds good prior to step (1). The complete range of β_0 in Eq. (40) equals

$$(6\pi - 2\zeta)\,(\sec\theta_0 - 1) \geqslant 5\pi\,(\sec\theta_0 - 1), \tag{45}$$

whereas the maximum change in β_0 due to step (1) is $|\Delta\beta|_{\max} = 2\pi\,(\sec\theta_0 - 1)$. Since the range of β_0 in Eq. (45) is more than $2|\Delta\beta|_{\max}$, we will select the appropriate direction of sweep (as shown in Fig. 6(b)) such that Eq. (40) is satisfied also after step (1). In step (2), we decrease θ to θ^* if $\theta > \theta^*$. If $\theta = \theta^*$, we change θ to θ_1, where θ_1 is given by

$$\begin{aligned}&\theta_1 = k_1\,\theta^*, \quad 1 < k_1 \leqslant k_{\max},\\ &\quad k_{\max} \triangleq (\pi/2-\varepsilon)/\theta^*_{\max} = (1.571-\varepsilon)/0.775.\end{aligned} \tag{46}$$

In Eq. (46), ε is arbitrarily small and k_1 is chosen so that the sphere does not end in a configuration with $n=\infty$ after Step (1).

$\underline{n=\infty}$: This occurs when $CO=0$, and $CF \neq 0$. As in $n=1$, we investigate the cases, $\theta_0 < \theta^*$, $\theta_0 = \theta^*$, and $\theta_0 > \theta^*$. If $\theta_0 < \theta^*$, we apply the TO maneuver. This changes n while θ_0 increases to θ^*. If $\theta_0 > \theta^*$, we reduce θ_0 to θ^* which again changes n. If $\theta_0 = \theta^*$, we increase θ to θ_1, where θ_1 is chosen as

$$\theta_1 = k_\infty\,\theta^*,$$

$$k_\infty \triangleq (\pi/2-\varepsilon)/\theta^*_{\max} = (1.571-\varepsilon)/0.775. \tag{47}$$

$\underline{n=0}$: In this case, $CF=0$, and $CO \neq 0$, as shown in Fig. 6(c). If $\beta_0 \neq 0$, we use a TO maneuver to increase θ_0 to θ^*. We choose to move C away from the origin to C_1, as illustrated in Fig. 6(c). This eliminates the possibility of ending in the special cases, $n=1$ or $n=\infty$.

If $\beta_0 = 0$, we will use control action (A) to increase the value of θ from zero to θ_1, defined as follows:

$$\theta_1 = \begin{cases} k_0\sqrt{x_0^2+y_0^2} & \text{if } k_0\sqrt{x_0^2+y_0^2} \leqslant (\pi/2-\varepsilon),\\ (\pi/2-\varepsilon) & \text{if } k_0\sqrt{x_0^2+y_0^2} > (\pi/2-\varepsilon). \end{cases} \tag{48}$$

This ensures that the maneuver does not end with $\theta > (\pi/2-\varepsilon)$. Also, action (A) is chosen to move C away from the origin along OF; this eliminates the possibility of ending with $n=1$ or ∞.

$\underline{n\text{ is undefined}}$: This case occurs when $CF = CO = 0$, and $\beta_0 \neq 0$. We first apply a TO maneuver to increase θ from zero to θ^*, as shown in Fig. 6(d), which results in an $n=1$ configuration. We then apply the steps outlined for $n=1$ to change n to a value in $(0,1)\cup(1,\infty)$.

$\underline{\theta_0 > (\pi/2-\varepsilon)}$: We apply action (A) to reduce θ to θ_1, where $\theta_1 \in [\theta^*_{\max}, (\pi/2-\varepsilon)]$ such that

$$\theta_1 = k_\theta\,\theta_0, \quad (\theta^*_{\max}/\theta_0) \leqslant k_\theta < 1, \tag{49}$$

k_θ is chosen so that configurations where n is 0, 1, ∞, or undefined, are avoided.

6.3. Complete reconfiguration algorithm

The complete reconfiguration strategy is explained using Fig. 7. In this diagram, all configurations of the sphere are divided into eight configuration sets, S_i, $i = 1, 2, \ldots, 8$, namely:

(1) S_1 : $\{\mathbb{X} \mid x = y = \theta = \beta = 0\}$ (equilibrium),
(2) S_2 : $\{\mathbb{X} \mid \theta \geqslant \pi/2\}$,
(3) S_3 : $\{\mathbb{X} \mid \theta < \theta^* < \pi/2,\ n \in (0,1)\cup(1,\infty)\}$,
(4) S_4 : $\{\mathbb{X} \mid \theta^* \leqslant \theta < \pi/2,\ n \in (0,1)\cup(1,\infty)\}$,
(5) S_5 : $\{\mathbb{X} \mid n = 0\}$,
(6) S_6 : $\{\mathbb{X} \mid n = 1\}$,
(7) S_7 : $\{\mathbb{X} \mid n = \infty\}$,
(8) S_8 : $\{\mathbb{X} \mid n$ undefined$\}$

and all possible transitions between them are considered based on Sections 6.1 and 6.2. In the description above, $\mathbb{X} = \begin{pmatrix} x & y & \theta & \alpha & \beta \end{pmatrix}^{\mathrm{T}} \in R^5$ denotes the configuration space of the sphere in accordance with Eq. (3). Theorem 7, which applies when $n \in (0,1)\cup(1,\infty)$ and $0<\theta\leqslant(\pi/2-\varepsilon)$, is shown by the transition $S_4 \to S_1$. When the initial configuration does not lie in $S_4 \cup S_1$, it belongs to the set $S_2\cup S_3\cup S_5\cup S_6\cup S_7\cup S_8$. Transition from these special cases to S_4 occur in finite time and require atmost two steps.

7. Stability analysis

7.1. Coordinate transformation

The reconfiguration algorithm guarantees that $(x, y, \theta, \beta) = (0, 0, 0, 0)$ is an equilibrium configuration to which all

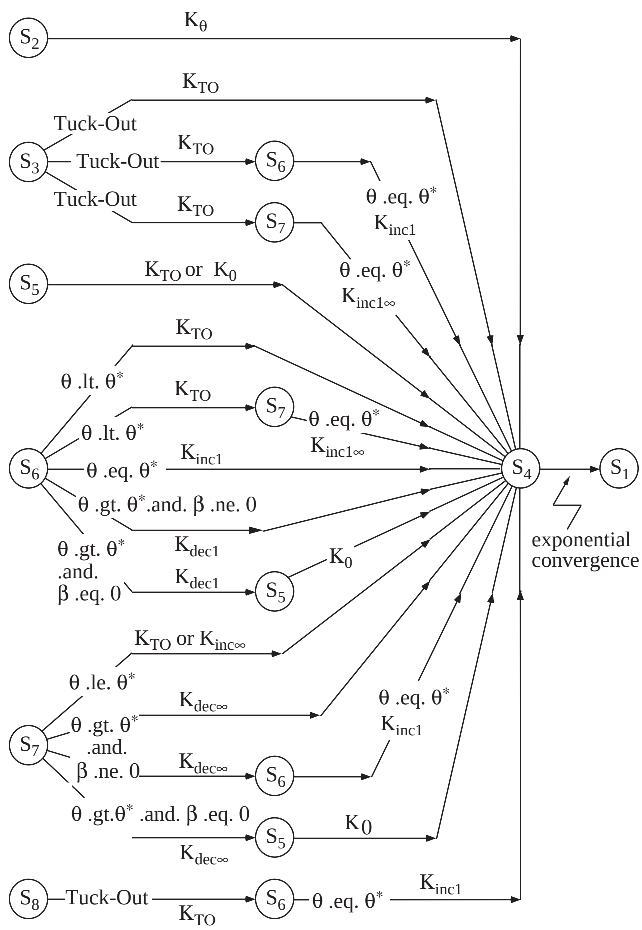


Fig. 7. Transition between all configuration sets.

other configurations are converged. We now prove that the equilibrium configuration is stable. To this end, we define

$$R = x^2 + y^2, \quad \Theta = \theta^2. \tag{50}$$

Since $(R, \Theta, \beta) = (0, 0, 0) \iff (x, y, \theta, \beta) = (0, 0, 0, 0)$, the equilibrium configuration can be defined as $(R, \Theta, \beta) \equiv (0, 0, 0)$. We will now prove stability of the equilibrium configuration by defining $\|\mathbb{X}\| = (R^2 + \Theta^2 + \beta^2)^{1/2}$ and showing that for each $\kappa > 0$ there exists a $\delta = \delta(\kappa)$ such that

$$\|\mathbb{X}(0)\| \leqslant \delta \quad \Rightarrow \quad \|\mathbb{X}(t)\| \leqslant \kappa. \tag{51}$$

Furthermore, we will prove exponential stability (Khalil, 2002) for all $\mathbb{X} \subset S_4$.

7.2. Exponential stability

We note that CF and CO decrease in geometric progression over each RS–DPT pair (see proof of Theorem 2). If we assume that each RS–DPT pair requires equal time to execute (this is conservative since the time required will be progressively less), CF and CO decrease in geometric progression over equal intervals of time. Such curves have exponential decay and we can write

$$\begin{array}{l} CF(t) = CF(0)\ \exp[-\gamma_1 t] \\ CO(t) = CO(0)\ \exp[-\gamma_1 t] \end{array} \quad \gamma_1 \triangleq \frac{1}{\Delta t}\ \ln\left(\frac{1}{r}\right), \tag{52}$$

where $CF(0)$ and $CO(0)$ are values of CF and CO at the beginning of the sweep-tuck algorithm, $t = 0$, and Δt is chosen conservatively as the time required for the first RS–DPT pair.

Since $CF = (\tan\theta - \theta)$ decreases exponentially in time, we can show that θ, $\tan\theta$, and $(\sec\theta - 1)$, decrease exponentially. Specifically, by differentiating the expressions of CF in Eqs. (9) and (52) and using the comparison Lemma (Khalil, 2002) we obtain

$$\theta(t) \leqslant \theta(0)\ \exp[-\gamma_1\gamma_2 t\,]$$

$$f[\theta(t)] \leqslant f[\theta(0)]\ \exp[-(\gamma_1/3)t\,] \quad f(\theta) \triangleq \tan\theta$$

$$g[\theta(t)] \leqslant g[\theta(0)]\ \exp[-(2\gamma_1/3)t\,] \ \ g(\theta) \triangleq (\sec\theta - 1), \tag{53}$$

where $\gamma_2 > 0$. We prove exponential stability of the equilibrium configuration next.

Theorem 8 (Exponential stability). *The reconfiguration algorithm in Theorem* 7 *renders the equilibrium configuration* $(R, \Theta, \beta) \equiv (0, 0, 0)$ *exponentially stable for* $\mathbb{X} \subset S_4$.

Proof. Consider an arbitrary configuration of the sphere in Fig. 2 for $\mathbb{X} \subset S_4$. Using the triangular inequality, the relation $CQ = \tan\theta$, and Eqs. (50), (52), and (53), we can write

$$R(t) \leqslant 2[CO^2(0) + CQ^2(0)]\ \exp[-(2\gamma_1/3)t\,]. \tag{54}$$

Using the triangular inequality we again write

$$CO^2(0) \leqslant 2[R(0) + CQ^2(0)]. \tag{55}$$

By substituting Eq. (55) into Eq. (54), we get

$$R^2(t) \leqslant 8[4R^2(0) + 9CQ^4(0)]\ \exp[-(4\gamma_1/3)t\,]. \tag{56}$$

Since $0 < \theta \leqslant (\pi/2 - \varepsilon)$, $CQ \leqslant c_1\sqrt{\Theta}$, where $c_1 \triangleq [\tan(\pi/2 - \varepsilon)/(\pi/2 - \varepsilon)]$. Thus, Eq. (56) yields

$$R^2(t) \leqslant 8[4R^2(0) + 9c_1^4\Theta^2(0)]\ \exp[-(4\gamma_1/3)t\,]. \tag{57}$$

From Theorem 7 we know

$$|\ \beta(t)\ | \leqslant (3\pi - \zeta)\,[\sec\theta(t) - 1] \leqslant 3\pi[\sec\theta(t) - 1].$$

Using Eq. (53) we can therefore write

$$\beta^2(t) \leqslant 9\pi^2[\sec\theta(0) - 1]^2\ \exp[-(4\gamma_1/3)t\,]. \tag{58}$$

Again, since $0 < \theta \leqslant (\pi/2 - \varepsilon)$, $\left[\sec\theta(t) - 1\right] \leqslant c_2\Theta$, where $c_2 \triangleq [\sec(\pi/2 - \varepsilon) - 1]/(\pi/2 - \varepsilon)^2$. Thus Eq. (58) yields

$$\beta^2(t) \leqslant (25/4)\,\pi^2 c_2^2\Theta^2(0)\ \exp[-(4\gamma_1/3)t\,]. \tag{59}$$

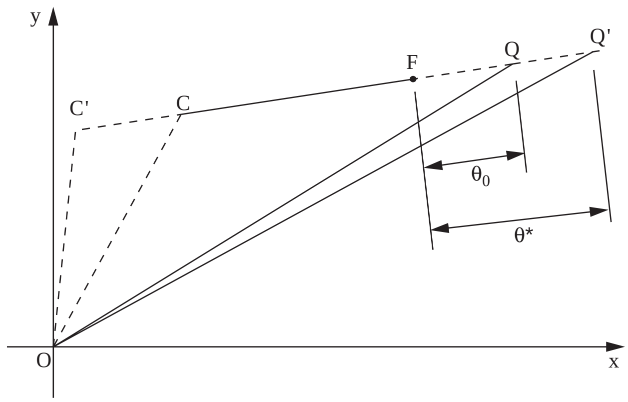


Fig. 8. Motion of the sphere during TO maneuver.

By combining Eqs. (57), (59), and the expression for $\theta(t)$ in Eq. (53), we get using Holder inequality

$$\begin{aligned}\|\mathbb{X}(t)\|^2 &= [R^2(t) + \Theta^2(t) + \beta^2(t)] \\ &\leqslant [c_3 R^2(0) + c_4 \Theta^2(0)] \exp[-\gamma_3 t] \\ &\leqslant c_5 \|\mathbb{X}(0)\|^2 \exp[-\gamma_3 t],\end{aligned} \tag{60}$$

where $c_3 \triangleq 32$, $c_4 \triangleq 72c_1^4 + (25/4)\pi^2 c_2^2 + 1$, $c_5 = \max[c_3, c_4]$, $\gamma_3 = 2\gamma_1 . \min[(2/3), \gamma_2]$. □

Theorem 8 proves exponential stability of S_1 for $\mathbb{X} \subset S_4$, which is similar to a local result. The stability of S_1 is established next.

Theorem 9 (Stability). *The reconfiguration algorithm in Fig. 7 guarantees uniform stability of the equilibrium configuration* $(R, \Theta, \beta) \equiv (0, 0, 0)$.

Proof. We provide an outline of the proof for conciseness. At the initial time, let $\mathbb{X} \subset S_3$. The sphere undergoes a TO maneuver whereby θ increases from θ_0 to θ^*, as shown in Fig. 8. For this maneuver, using the triangular inequality

$$\sqrt{R(t)} \leqslant \sqrt{R(0)} + (\theta^* - \theta_0) \leqslant \sqrt{R(0)} + \theta^*. \tag{61}$$

Also, from Eq. (43) we know

$$|\beta_0| = 5\pi/2(\sec\theta^* - 1) \;\Rightarrow\; \theta^2 \leqslant \theta^{*2} \leqslant (4/5\pi)|\beta_0|. \tag{62}$$

Since a TO maneuver does not change the value of $\beta = \beta_0$, we can show using Eqs. (61) and (62).

$$\|\mathbb{X}(t)\| \leqslant K_{TO} \|\mathbb{X}(0)\|, \quad K_{TO} \triangleq 2\sqrt{2}. \tag{63}$$

The constant K_{TO} is shown in Fig. 7 for transitions from S_3 to S_4, S_6, and S_7. The constants associated with transition from other configuration sets are similarly derived (Das, 2002) based on discussion in Section 6. All of them are greater than unity and are shown in Fig. 7. For transition from $S_2 \cup S_3 \cup S_5 \cup S_6 \cup S_7 \cup S_8$ to S_4 we can therefore claim

$$\|\mathbb{X}(t)\| \leqslant K_{\max} \|\mathbb{X}(0)\|. \tag{64}$$

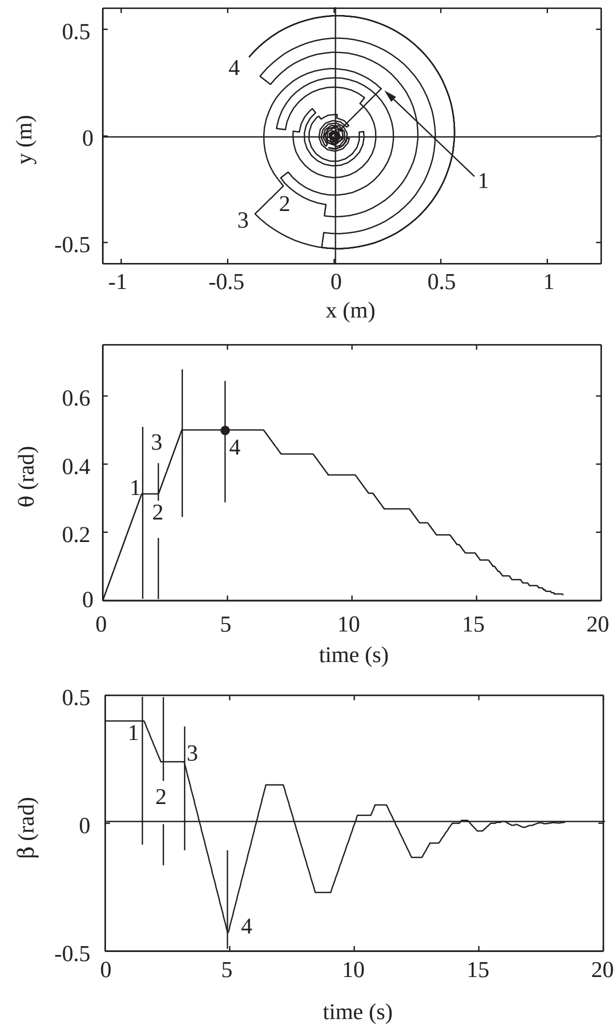


Fig. 9. Simulation of complete reconfiguration.

$K_{\max} \triangleq \max\{K_\theta, K_{TO}K_1, K_O K_2, K_{\mathrm{inc}1} K_{\mathrm{dec}\infty}\}$, $K_1 \triangleq \max\{K_{\mathrm{inc}1}, K_{\mathrm{inc}\infty}\}$, $K_2 \triangleq \max\{K_{\mathrm{dec}1}, K_{\mathrm{dec}\infty}\}$.

For the transition $S_4 \to S_1$, we have from Eq. (60)

$$\|\mathbb{X}(t)\| \leqslant \sqrt{c_5} \|\mathbb{X}(0)\|. \tag{65}$$

By combining Eqs. (64) and (65), we get

$$\|\mathbb{X}(t)\| \leqslant K_{\max} \sqrt{c_5} \|\mathbb{X}(0)\|. \tag{66}$$

Thus, for any $\kappa > 0$, we can choose $\delta = (1/K_{\max}\sqrt{c_5})\kappa$ such that Eq. (51) is satisfied. □

8. Simulation results

We assume the following initial configuration:

$$(x \;\; y \;\; \theta \;\; \alpha \;\; \beta) \equiv (0.0 \;\; 0.0 \;\; 0.0 \;\; \pi/4 \;\; 0.4), \tag{67}$$

where the units are in meters and radians. The simulation result, shown in Fig. 9, is representative of the special cases $n=$ undefined and $n = 1$, and the general case $n \in (1, \infty)$. Since

$n=$ undefined at the initial time, the sphere first performs a TO maneuver—this results in the configuration marked 1. Thereafter, the sphere acquires a configuration where $n=1$ and $\theta=\theta^*=0.31$. In conformity with our discussion in Section 6.2, the sphere now sweeps to align O, C and F and reaches the configuration marked 2. Subsequently, it increases θ to θ_1 defined in Eq. (46) for the choice of $k_1=1.6$. This changes θ to 0.5, n to 1.84, and the configuration to the point marked 3. A PPS maneuver is now performed and the sphere moves to the configuration marked 4. With $\psi=0.65\in[\max(\bar{\psi},\Psi),\ \cos^{-1}(1/n)]$, complete reconfiguration is achieved with a sequence of CRS–DPT pairs.

9. Conclusion

The stabilization problem of the rolling sphere has eluded a solution since its kinematic model cannot be reduced to chained-form. We present a discontinuous control strategy for exponential stabilization of an arbitrary desired configuration. Our design is based on a specific choice of Euler angle description of sphere orientation, placement of the desired orientation at the singularity of the representation, and choice of control inputs in a rotating reference frame. Our control inputs result in sweep and tuck maneuvers of the sphere. A sequence of these maneuvers result in global stability of the equilibrium configuration and exponential convergence to it from a large and well-defined set in the configuration space. A few preliminary maneuvers are required for configurations lying outside the set to be converged within the set—these maneuvers require finite time and maintain the notion of stability.

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