Control-Oriented Design using $\mathcal{H}_\infty$ Synthesis and Multiple Surrogates

Sanketh Bhat∗, Felipe A. C. Viana†, Rick Lind‡ and Raphael T. Haftka§

University of Florida

This paper discusses a ‘structure/control’ simultaneous design approach with the introduction of frequency-dependent weighting functions in the $\mathcal{H}_\infty$-control synthesis framework as design variables. This control-oriented design approach minimizes closed-loop metrics for optimal performance by searching over the design space encompassing the open-loop dynamic, weighting function and controller variables, while maintaining constraints for controller existence. Hence, the optimal design does not need to compute both the open-loop plant and the controller simultaneously, instead, the design simply finds the open-loop dynamics for which a controller exists that achieves the lowest closed-loop metrics. So it is the ‘structural design/controller existence’ problem that that is being tackled. This methodology has been implemented for vibration-attenuation in hypersonic vehicles. This paper uses multiple surrogates along with NSGA-II for optimization.

I. Introduction

The performance of many systems is evaluated through closed-loop metrics; consequently, many concepts have been investigated to design systems that optimize these closed-loop metrics. This concept of integrated design has particularly been considered for optimization of a structure and a controller.1–19 The actual metrics that relate performance can often be cast as norms on the transfer functions from commands to responses; however, some metrics must be represented by non-norm metrics. Metrics for handling qualities of aircraft are particularly problematic for design since they are a mix of norm and non-norm metrics.20

An approach was recently formulated that introduced robust control, using metrics such as $\mathcal{H}_\infty$, for integrated design.21, 22 This approach replaced the explicit dependence on the controller gains with a set of implicit conditions that note controller existence. In this way, the resulting design computed a structure for which a controller existed that minimized a closed-loop norm. That approach was able to generate an optimal controller that minimizes a particular closed-loop norm; however, it is limited to consideration of norms as the closed-loop metrics.

This paper introduces an extension to the existence-based formulation that allows for non-norm metrics. The fundamental concept introduces variables to the design space that represent weightings on the errors of the closed-loop transfer function. These weights are determined by desired characteristics for the closed-loop transfer function and its associated norm; however, it is unclear how to determine these weights for non-norm metrics of that closed-loop transfer function. As such, the introduction of the weights in the design process allows an optimal $\mathcal{H}_\infty$ controller to be synthesized that minimizes a non-$\mathcal{H}_\infty$ metric.

This methodology has been proposed for hypersonic vehicles which mitigates vibrations due to the undesired thermal effects on the vehicle.21, 22 The issue of vibration attenuation actually considers both the thermal profile and associated structural dynamics of the vehicle along with a controller and associated weightings that eliminates any bending. In this sense, the design is a multi-disciplinary design of both a structure and a controller.

The remainder of the paper is organized as follow. In Section II the control-oriented design methodology is discussed, specifically, the design space is introduced, the optimization problem is formulated and the
theory of $H_{\infty}$-synthesis and surrogate-based optimization using NSGA-II is discussed. The case-study of the hypersonic vehicle is used to illustrate this methodology in Section III.

II. Structure/Control Simultaneous Design

A. System Representation

A generalized description of a closed-loop system is presented in Figure 1 as a feedback relationship between operators. The open-loop model is given as $P$ and contains the nominal estimates of the dynamics for the system. The feedback compensator is given as $K$ and contains the relationship between sensor measurements and actuator commands. Another element, $\Delta$, represents the uncertainty associated with errors and unknown values of the open-loop dynamics. The remaining element, $W$, represents the weighting functions that normalize the errors. The closed-loop operation of such systems suggest that the design may be decomposed into separate subspaces.

![Figure 1. Closed-loop model](image)

Part of the variables that compose the design space are related to the open-loop dynamics. This space, defined as $\mathcal{P}$, can include a wide variety of variables including geometry, structure, materials and other aspects related to vehicle design. A particular configuration of the open-loop dynamics is thus represented by the vector, $\pi \in \mathcal{P}$, within the design space.

Part of the variables that compose the design space are related to the elements that may be varied. This space, defined as $\mathcal{K}$, can include aspects of the feedback compensator such as gains or variables used to select state-space matrices. Any controller is thus formulated using the vector, $\kappa \in \mathcal{K}$, from within the design space.

The remaining variables considers the weighting functions that may be varied. This space, defined as $\mathcal{W}$, can include poles and zeros of transfer functions along with gains. A specific set of weightings that normalizes the errors is associated with a vector, $\omega \in \mathcal{W}$, within the design space.

B. Design

The approach for control-oriented design is expressed as minimizing closed-loop metrics, given as $f \in \mathbb{R}^m$, by searching over the design space, given as $\mathcal{P}$ and $\mathcal{W}$, while maintaining the feasibility constraints for controller existence as in Equation 1, where $m$ is the number of objective functions.

\[ \| W T_{de} \| \leq 1 \]  
\[ \| W \| = 1/\varepsilon. \]

\[ a \] It is much more convenient to reflect the system performance objectives by choosing appropriate weighting functions. Some components of a signal may be more important than others and not all the signals may be measured in the same units. Weighting functions essentially, can be used to make such signal components comparable. Also, frequency-dependent weights help to reject errors in the desired frequency range. The weighting functions in Figure 1 is chosen to reflect the design objectives and knowledge of the disturbance and sensor noise. For example, the performance weighting may be used to reflect requirements on the shape of closed-loop transfer functions and the actuator weighting may be used to reflect limits on the actuator outputs. A specific set of weightings that normalizes the errors is associated with a vector, $\omega \in \mathcal{W}$, within the design space.
\[
\min_{\pi \in \mathbb{P}} f(\pi, \omega, \kappa) \quad (1)
\]

\[
\omega \in \mathbb{W} \\
\kappa \in \mathbb{K} 
\]

such that

\[
0 = XA(\pi, \omega) + A(\pi, \omega)^*X + X(\frac{1}{\gamma^2}B_1(\pi, \omega)B_1(\pi, \omega)^* - B_2(\pi, \omega)B_2(\pi, \omega)^*)X + C_1(\pi, \omega)^*C_1(\pi, \omega) \quad (2)
\]

\[
0 = A(\pi, \omega)Y + YA(\pi, \omega)^* + Y\left(\frac{1}{\gamma^2}C_1(\pi, \omega)^*C_1(\pi, \omega) - C_2(\pi, \omega)^*C_2(\pi, \omega)\right)Y + B_1(\pi, \omega)B_1(\pi, \omega)^* \quad (3)
\]

\[
\gamma^2 > \rho(XY) \quad (4)
\]

Control synthesis is a process of choosing a controller, \(K\) such that certain weighted signals are made small. This research takes advantage of the well-established theory of \(H_\infty\) control synthesis to define the smallness metric using the \(H_\infty\)-norm criteria. \(H_\infty\)-norm, \(\gamma\) is a closed-loop performance metric that is defined as \(|W_F(P, K)|_\infty\) on the transfer function from disturbances, \(d\) to errors, \(e\). The objective of \(H_\infty\)-norm synthesis is to find a controller, \(K\) to stabilize the closed-loop system given in Figure 1 and to minimize \(\gamma\). If \(\gamma \leq 1\), the system is said to achieve robust performance or stabilization with respect to both parametric and dynamic uncertainties in \(\Delta\).

The controller existence conditions are expressed as the pair of Riccati equations in Equation 2 and Equation 3 along with the constraint of spectral radius (\(\rho\)) in Equation 4. The open-loop (\(\pi\)) and weighting function (\(\omega\)) variables are introduced through the dynamics of the open-loop synthesis model, \(A(\pi, \omega), B_i(\pi, \omega)\) and \(C_i(\pi, \omega)\) where \(i = 1, 2\). The open-loop synthesis model is the model without the controller, \(K\) in Figure 1.

The elements, \(\kappa\) of \(\mathbb{K} = \{X, Y | X = X^* > 0, Y = Y^* > 0\}\) are the solutions to the Riccati equations needed for controller existence while the elements of \(\mathbb{P}\) correspond to the open-loop model. The introduction of \(\mathbb{W}\), represents the variables of the frequency-dependent weighting functions. The values of \(\kappa\) are implicitly solved using some in-built functions in Matlab.\(^b\)

In this way, the optimal design actually does not need to compute both the open-loop plant, \(P\) and controller, \(K\) simultaneously; instead, the design can simply find the open-loop dynamics for which a controller exists that achieves the lowest closed-loop metrics \((f)\). This issue of existence, or feasibility, is the central issue that enables closed-loop design. The design problem can be rephrased so as to find the best open-loop design variables, \((\pi)\), the controller design variables, \((\kappa)\) and the weighting function design variables \((\omega)\) for which a controller, \(K\) exists and optimizes the closed-loop mission performance. This ‘structural design/controller existence’ problem being handled overcomes some of the drawbacks of the sequential approach by enabling optimal choice of sensors and actuators and ensures that mission requirements are met and at the same time does not make the problem computationally intractable.

\[\text{C. Surrogate-based optimization using NSGA-II}\]

Design optimization requires a large number of expensive simulations/experiments. To reduce the computational cost, cheap-to-evaluate surrogate models (also known as meta-models) are often used in place of the actual simulation models.\(^{25-28}\) That is, the expensive-to-evaluate response \(y(x)\) is approximated by a cheaper model \(\hat{y}(x)\) based on (i) assumptions on the nature of \(y(x)\), and (ii) on the observed values of \(y(x)\) at a set of \(p\) data points called experimental design. More explicitly it is given in Equation 5, where \(x = [x_1, ..., x_d]^T\) is a real \(d\)-dimensional vector and \(\epsilon(x)\) represents both the error of approximation and measurement (random) errors.

\(^b\)The case study in this paper will take advantage of some MATLAB in-built functions like \textit{hinfsyn}()\(^{24}\) to solve for the controller existence conditions. The controller existence condition arises when noting that synthesis of \(H_\infty\)-norm controllers actually follow a two-step procedure. The initial step iterates over the controller existence conditions given in Equation 2, Equation 3 and Equation 4 that indicates if a controller exists to achieve a particular value of closed-loop norm, \(\gamma\) and solves for \(X\) and \(Y\). The final step computes the gain for the feedback controller that achieves the optimal closed-loop norm, \(\gamma_{opt}\). This two-step procedure is implemented in professional software, such as MATLAB. So this study does not explicitly solve for the controller variables, \(\kappa\) i.e. \(X\) and \(Y\).
\[ y(x) = \hat{y}(x) + \varepsilon(x) \] (5)

\[ e_{RMS} = \sqrt{\frac{1}{V} \int_V e^2(x) dx} = \sqrt{\frac{1}{V} \int_V [\hat{y}(x) - y(x)]^2 dx} \] (6)

The accuracy of a surrogate is measured by the root mean square error, \( e_{RMS} \) shown in Equation 6. Due to the computational cost of estimating \( e_{RMS} \), cross validation is often used as an alternative for both assessing accuracy and surrogate selection. It is attractive because it does not depend on the statistical assumptions of a particular surrogate technique and it does not require extra test points.\(^{29,30}\) Cross validation is a process of estimating errors by constructing the surrogate without some of the points and calculating the errors at these omitted points. The process proceeds by dividing the set of \( p \) data points into \( k \) subsets. The surrogate is fit to all subsets except one, and error is computed for the omitted subset that was left out. This process is repeated for all subsets to produce a vector of cross-validation errors \( e_{XV} \) (also known as the vector, where \( X \) stands for prediction sum of squares). Figure 2 illustrates the cross-validation errors for a kriging surrogate. The \( e_{RMS} \) is estimated from \( e_{XV} \) as shown in Equation 7.

\[ PRESS_{RMS} = \sqrt{\frac{1}{p} e_{XV}^T e_{XV}} \] (7)

![Figure 2. Cross-validation error \( e_{XV} \) at the second point of the experimental design exemplified by fitting a kriging model (KRG) to \( p = 5 \) data points](image)

Surrogate-based optimization evolves in a cycle that consists of analyzing a number of designs, fitting a surrogate, performing optimization based on the surrogate, and finally performing exact simulation at the design obtained by the optimization. This work takes advantage of multiple surrogates when performing optimization. Several surrogates are fit to the simulation codes instead of one. Here, the traditional polynomial response surface\(^{31,32}\) kriging\(^{33,34}\) and radial basis neural network models\(^{35,36}\) are used. Then, the optimization is solved several times, each one using different surrogates for the objectives. The authors believe that the use of multiple surrogates is beneficial because fitting many surrogates and repeating optimizations is cheap compared to cost of simulations and most accurate surrogate does not guarantee the global optimum.\(^{37,38}\)

In Equation 1, the design problem is to simultaneously optimize several objectives, it is expected that no single design will simultaneously perform best for all metrics, and so a trade-off curve (Pareto optimal front) between the metrics is usually preferred. In this paper, the multi-objective optimization is handled using the elitist non-dominated sorting genetic algorithm\(^ {39,40}\) (NSGA-II). As in many implementations of genetic algorithm, the design variables are discretized into a finite number of intervals. Then, NSGA-II ranks designs by non-domination (instead of by values of the objective functions). Low-rank designs are given the greatest probability of being selected by the operators of the genetic algorithm (selection, crossover, mutation). Constraints are handled by constraint-domination; that is, if one design is feasible and the other is not, the former is obviously favored. If both designs are infeasible, the design with a smaller overall constraint violation is favored. Surrogates are created for objectives and constraints and then the NSGA-II is run using
different surrogates for the objectives. With that the authors hope to populate the pareto front faster than if a single surrogate per objective is used.

III. Hypersonic Vehicle

A. Goal

The concept of control-oriented design is demonstrated for a hypersonic vehicle. The mission is prescribed by changing airspeed and altitude. However, several difficulties must be circumvented for this maneuver. The propulsion system is tightly coupled to the structural dynamics of the fuselage so vibrations can cause loss of engine performance. The vibrations are compounded by the introduction of thermal gradients which result from the tremendous heating across the fuselage throughout flight. As such, vibration attenuation becomes a critical aspect of mission performance.

The objective of this case study is to find the operating range, \( \pi \in \mathbb{P} \) (open-loop variables) to design the thermal protection system along with the optimal weightings, \( \omega \in \mathbb{W} \) (weighting function variables) for which a controller exists \( b, \kappa \in \mathbb{K} \) (controller variables) that minimizes the vibrations on the vehicle. A model-following approach is used, where the closed-loop system is expected to emulate a reference model with desired closed-loop properties. This case-study considers \( \Delta = 0 \), i.e. there is no uncertainty associated with the errors and system variables.

B. Vehicle

A representative model is used in the paper that describes the flight dynamics for a baseline configuration.\(^{41-46}\) This reduced-order model has 5 rigid-body states corresponding to the \( V \) for forward velocity, \( \alpha \) for angle of attack, \( h \) for altitude, \( q \) for pitch rate and \( \theta \) for pitch angle. Additionally, it contains 6 flexible-body states corresponding to 3 bending modes for the fuselage; however, only the first-bending mode is included in this paper since the effects of the higher modes are negligible on the flight dynamics. The dynamics include responses to 4 control effectors given as elevator angle (\( \delta_e \)), canard angle (\( \delta_c \)), diffuser-area ratio (\( \delta_d \)) and fuel equivalence ratio (\( \delta_f \)).

The equations of motion are formulated in Equations 8-13 for the nonlinear dynamics.\(^{44}\) This formulation uses \( m \) as the vehicle mass, \( I_{yy} \) as the moment of inertia, \( g \) as the acceleration due to gravity, \( T_h \) as thrust, \( D \) as drag, \( L \) as lift, \( M \) as the pitching moment, \( \xi \) as structural damping and \( \omega \) as natural frequency of first the bending mode along with \( N \) as the generalized elastic force. A linearized model is obtained from perturbations about a trim condition for this model. The expressions for the forces and moments used to trim the vehicle are lengthy so are omitted for brevity.\(^{41}\)

\[
\begin{align*}
\dot{V} &= \frac{T_h \cos \alpha - D}{m} - g \sin \theta - \alpha \\
\dot{h} &= V \sin \theta - \alpha \\
\dot{\alpha} &= -\frac{L + T_h \sin \alpha}{mV} + q + \frac{g}{V} \cos \theta - \alpha \\
\dot{\theta} &= q \\
\dot{q} &= \frac{M}{I_{yy}} \\
\ddot{\eta} &= 2\xi \omega \dot{\eta} - \omega^2 \eta + N
\end{align*}
\]

A primary characteristic of this vehicle affecting the dynamics is the integrated fuselage and propulsion system as shown in Figure 3. The fuselage is actually designed to be part of the engine system by using the forebody as a compressor and the aftbody as an external nozzle. This design introduces a significant amount of coupling between the aerodynamics and propulsion dynamics. Firstly, the airflow across the forebody compressor introduces a lift force and a nose-up pitching moment while the airflow through the external nozzle introduces a lift force and a nose-down pitching moment so variations in propulsion performance alter the aerodynamic characteristics. Conversely, any variation in angle of attack and sideslip affects the engine inlet conditions so the propulsion performance is altered by variations in aerodynamic characteristics. Also,
the bending structure changes the flow turn angle which changes the inlet conditions which in turn changes the thrust angle so there is an especially strong and fast coupling between pitch and propulsion.

Figure 3. Air-breathing hypersonic vehicle, http://mix.msfc.nasa.gov/abstracts.php?p=2468, reprinted with permission

A typical mission for this vehicle is to place some payload into low Earth orbit which requires the vehicle to operate in many flight regimes such as subsonic, transonic, supersonic, hypersonic and orbital. Each regime introduces control problems that must be alleviated for a successful mission. For example, the control surfaces will probably be small so as to minimize heating during hypersonic flight, but this may create difficulties for properly controlling the vehicle at low supersonic speeds. Another potential control problem may arise from the shocks generated by unsteady aerodynamics at transonic flight. Also, the issue of orbit transfers for payload delivery while in space is a control problem for this type of vehicle that introduces issues not usually affecting atmospheric flight. The control problems in every flight regime are important; however, this paper will limit consideration to the hypersonic regime while still in the atmosphere.

C. Control Synthesis Architecture

The effects of thermal gradients on the flight dynamics should be minimized; however, the rigid-body maneuvering should not be altered. As such, the control objective becomes vibration attenuation using a controller that does not significantly use low-frequency energy near the rigid-body modes. This objective allows the resulting controller to fit within a multi-loop scheme involving an inner-loop controller for vibration attenuation and a separate outer-loop controller for maneuvering and guidance. This paper considers only the design of the inner-loop controller.

A stabilizer, \( S \), is initially designed for the open-loop dynamics prior to synthesizing the vibration-attenuation controller. This element simply stabilizes the rigid-body dynamics by low-frequency actuation. Such a stabilizer is required because the synthesis for vibration attenuation must stabilize the closed-loop system; however, the vibration-attenuation controller should not affect the rigid-body dynamics. This initial stabilizer ensures the synthesis for vibration attenuation does not need to utilize any low-frequency actuation to affect the rigid-body modes. It uses measurements of pitch rate \( q \) to determine commands for the elevator and canard to achieve this stabilization. The stabilizer, \( S \) is created using \( H_\infty \) synthesis.

A target model, \( T \), is formulated that represents an acceptable set of dynamics. The rigid-body dynamics are identical to the hypersonic model but the structural mode has higher damping (\( \xi \)) than the hypersonic model. This increase in damping results in a considerable difference, as shown in Figure 4(a) and Figure 4(b), between the transfer functions from elevator to pitch rate. Note the peak near 0.04 rad/s is associated with a rigid-body flight mode while the peak near 22 rad/s is associated with the structural mode that should be attenuated. The output of \( T \) is the desired pitch rate, \( q_d \). As can be seen from Figure 4(c) and Figure 4(d), the target time response shown for 200 sec has subdued transient behavior due to increased damping compared to the nominal dynamics.

The block diagram used to synthesize the controller for vibration attenuation is given in Figure 5. The synthesis model relates the open-loop dynamics, \( P \) to a set of weighted errors, \( e \) and disturbances, \( d \). The input to the controller is \( y_q \) which is a noisy measurement of error in pitch rate while the commands from the controller are \( u_e \) and \( u_c \) as actuation for the elevator and canard. The outer-loop controller, which generates commands for maneuvering and guidance, provides the values of \( \{ \delta_e, \delta_c, \delta_d, \delta_f \} \) as the elevator, canard, diffuser-area ratio, and fuel equivalence ratio.
A model-following approach is adopted such that an error, $e_q$, is formulated as the difference between the measured pitch rate ($q$) and the desired pitch rate ($q_d$). This error is normalized to reflect the performance goals across frequency using a weighting, $W_q = K_1 \frac{s + a_1}{s + a_2}$, on the difference in pitch rate. The performance objective is generally to have good tracking at low frequencies and noise attenuation at high frequencies. So $W_q$ needs to normalize the errors, such that the tracking errors are small at low frequencies, but should not track noise at high frequencies. Hence, $W_q$ is a low-pass filter.

Additional errors are generated as penalties on the actuation. The error of $e_c$ is defined for the penalty on elevator and $e_c$ is defined for the penalty on canard. These errors are normalized by scaling the controller commands through the diagonal matrix of $W_k = K_2 \frac{s + b_1}{s + b_2}$. This weighting allows high actuation at low frequencies (for good tracking) but low actuation at high frequencies (to avoid tracking high frequency noise) for either the elevator or canard. Hence, $W_k$ is a high-pass filter.

Noise, $n$, is associated with the sensor measurement of pitch rate. This signal is a unity-bounded random signal that is weighted through $W_n = 0.01$ to limit the relative size of this noise in comparison to the pitch rate.

Also, a filter of $F = \frac{1000}{s + 1000}$ is included to eliminate the effective dependence of the input matrix on time. Such a filter does not alter the results in any appreciable fashion; however, it satisfies assumptions required by the synthesis algorithms.

The resulting controller is synthesized to minimize the gain, $\gamma$ from disturbances to errors. In this sense,
the weighting functions such as $W_q$ and $W_k$ serve to normalize the signals and reflect the desired performance as a function of frequency. The controller needs to add damping ($\xi$) to the system to achieve the objective of vibration-attenuation.

D. Optimization Problem Formulation

The design space consists of $\mathbb{P}$ which contains variables for the fuselage structure and the thermal protection system, $\mathbb{W}$ for the weighting function variables along with $\mathbb{K}$ which contains variables for a feedback controller. It is important to note that the geometry is relatively fixed due to aerodynamic issues while the thermal issues and structural components have considerable freedom in their design. In this case, the design space is appropriate since the thermal protection system and structure interact to determine the vibration characteristics of the fuselage along with associated heating effects. The design variables are listed in Table 1.

![Figure 5. Open-loop synthesis model used to create the controller, $K(\kappa)$. The dynamics of the synthesis model are the $A(\pi, \omega), B_i(\pi, \omega)$ and $C_i(\pi, \omega), i = 1, 2$ in Equation 2 and Equation 3](image)

<table>
<thead>
<tr>
<th>No.</th>
<th>Design Variable</th>
<th>Type</th>
<th>Bounds</th>
<th>Constraints</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$T_{nose}$</td>
<td>Open loop ($\pi$)</td>
<td>$[450, 900]$</td>
<td></td>
<td>Nose is the hottest part</td>
</tr>
<tr>
<td>2</td>
<td>$T_{tail}$</td>
<td>Open loop ($\pi$)</td>
<td>$[100, 800]$</td>
<td>$T_{nose} \geq T_{tail} + 100$</td>
<td>Tail is the coldest part</td>
</tr>
<tr>
<td>3</td>
<td>$K_1$</td>
<td>Weighting function ($\omega$)</td>
<td>$[0.1, 1]$</td>
<td>$K_1 &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$K_2$</td>
<td>Weighting function ($\omega$)</td>
<td>$[0.75, 1.25]$</td>
<td>$K_2 &gt; 0$</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$a_1$</td>
<td>Weighting function ($\omega$)</td>
<td>$[75, 125]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$a_2$</td>
<td>Weighting function ($\omega$)</td>
<td>$[4, 6]$</td>
<td>$a_1 &gt; a_2$</td>
<td>$W_q$ is a low-pass filter</td>
</tr>
<tr>
<td>7</td>
<td>$b_1$</td>
<td>Weighting function ($\omega$)</td>
<td>$[8, 12]$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$b_2$</td>
<td>Weighting function ($\omega$)</td>
<td>$[80, 120]$</td>
<td>$b_1 &lt; b_2$</td>
<td>$W_k$ is a high-pass filter</td>
</tr>
</tbody>
</table>

The design space for the open-loop dynamics consists of a 2-dimensional set, $\mathbb{P}$, related to effective temperature, $\pi = [T_{nose}, T_{tail}]$. In this case, a set of thermal profiles are chosen that have constant gradient.

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The open-loop, $\pi$ and weighting, $\omega$ variables affect the creation of the controller, $K$ from Equation 2, Equation 3 and Equation 4 and the block diagram shown in Figure 5.
from the nose to tail. This set, as shown in Figure 6(a), considers variations in both the tail temperature and nose temperature with the restriction that the nose must be at least 100°F hotter than the tail. This restriction is consistent with computational fluid dynamics and computational structural dynamics analysis on the hypersonic vehicle that the nose is hottest part of the structure because of shock waves. The open-loop dynamics, \( P(\pi) \) are parametrized as function of these effective temperatures to reflect variations in the Young’s modulus at the nose and tail which result from the structural elements and thermal protection system. As shown in Figure 6(b), the fuselage is split into 9 equal sections with each section having a constant thermal gradient and hence constant structural properties like Young’s modulus \((E_n)\).

![Figure 6. Thermal gradients for the hypersonic vehicle](image)

The design space for the weighting functions consists of a 6-dimensional set, \( W \) to reflect the performance and controller actuation weighting elements, \( \omega = [K_1, K_2, a_1, a_2, b_1, b_2] \). The elements basically reflect the region where performance is critical and actuation energy needs to be used.

The goal of feedback control design is to develop a controller to track a reference output i.e. minimize the error between the target and actual trajectory by utilizing the minimum energy possible. The objective can be rephrased so as to optimize performance by minimizing some error \((e_p)\) between the actual and desired response by utilizing the minimum actuation \((c)\) possible along with maintaining the closed-loop \( H_\infty \)-norm, \( \gamma \) to be \( \leq 1 \) to guarantee robustness with respect to uncertainties. This example can be extended to include uncertainties, i.e for \( \Delta \neq 0 \) but here the constraint guarantees performance. The constrained optimization can be stated as in Equation 14 where \( e_p \) is the error metric for performance, \( c \) is the control power utilized and \( \gamma \) is the closed-loop \( H_\infty \) norm. Since a model-matching approach is followed to minimize the error between the actual pitch rate \((q)\) and the desired pitch rate \((q_d)\) using the elevator as the control surface, the performance metric chosen is the root mean square error \((e_{RMS})\) between \( q \) and \( q_d \). The controller power metric is the mean square (MS) of the actuation used, i.e. elevator deflection \((\delta_e)\) time history.

\[
\min_{\omega \in W} \left\{ \begin{array}{l} e_p \\ c \end{array} \right\} \text{ such that } \gamma \leq 1 \tag{14}
\]

The objectives chosen are shown in Equation 15 where \( e_p \) and \( c \) are the normalized values of \( p_{RMSE} \) and \( c_{MS} \) respectively. The time response is observed to have steady-state errors in \( q \), so the performance objective considered has both steady-state and transient components and is considered for \( T = 200 \text{ sec} \).
\[
pr_mse = \sqrt{\frac{1}{T} \int_0^T (q - q_d)^2 dt}
\]

\[
c_m = \frac{1}{T} \int_0^T \delta^2 dt
\]

E. Results and Discussion

The process is started by sampling the design space with 700 points (this is enough to have approximately three points per orthant) using Latin hypercube sampling. This initial set of data is used to fit four different surrogate modeling techniques, shown in Table 3. The DACE toolbox of Lophaven et al. and the native neural networks MATLAB toolbox were used to generate the kriging and radial basis neural network models. The SURROGATES toolbox of Viana was used to run the polynomial response surface algorithm and it was also used for easy manipulation of the surrogates.

Table 2. Information about the set of surrogates used in the surrogate-based optimization

<table>
<thead>
<tr>
<th>Surrogates</th>
<th>Details</th>
</tr>
</thead>
<tbody>
<tr>
<td>KRG</td>
<td>Kriging model: Constant trend function and Gaussian correlation</td>
</tr>
<tr>
<td>PRS2 and PRS3</td>
<td>Polynomial response surface: Second and third order full models</td>
</tr>
<tr>
<td>RBNN</td>
<td>Radial basis neural network: Goal = (0.05\overline{g})^2 and Spread = 1/3</td>
</tr>
</tbody>
</table>

The estimated accuracy of the set of surrogates is first studied as shown in Table 2. Table 3 gives the \%PRESS\textsubscript{RMS} for the set of surrogates. As the \%PRESS\textsubscript{RMS} in the \(c\) and \(\gamma\) surrogate fits is less than 10, the surrogate fits are a good approximation to the actual data set.

Table 3. \%PRESS\textsubscript{RMS}(PRESS\textsubscript{RMS}/Range\%) values for different surrogate models for the initial configuration. Range: \(e_p = [0,1]\), \(c = [0,1]\) and \(\gamma = [0,45]\) (\(e_p\) and \(c\) are the normalised values of \(pr_mse\) and \(c_m\) in Equation 15)

<table>
<thead>
<tr>
<th>Metric</th>
<th>KRG</th>
<th>PRS2</th>
<th>PRS3</th>
<th>RBNN</th>
</tr>
</thead>
<tbody>
<tr>
<td>(e_p)</td>
<td>13</td>
<td>13</td>
<td>13</td>
<td>16</td>
</tr>
<tr>
<td>(c)</td>
<td>10</td>
<td>8</td>
<td>8</td>
<td>13</td>
</tr>
<tr>
<td>(\gamma)</td>
<td>4</td>
<td>4</td>
<td>4</td>
<td>4</td>
</tr>
</tbody>
</table>

The two surrogates giving the lowest \(PRESS\textsubscript{RMS}\) values i.e KRG and PRS for the objective, \([e_p, c]\) and constraint, \(\gamma\) is given to the NSGA algorithm. Figure 8 shows the evolution of the pareto front with multiple surrogates. There are new points being added to the pareto front at every iteration i.e. some of the points which were originally on the pareto front are no longer on the final pareto front. There are a total of 28 points on the final pareto front out of 1100 points. Also, the values go slightly negative in the control MS axis, which shows that optimization does indeed result in better pareto front. The Appendix illustrates the optimization process using only polynomial surface fits and with multiple surrogates. The benefit of using multiple surrogates is evident from the analysis shown in the Appendix.

The \(\Diamond\) on the pareto front in Figure 9 corresponds to the maximum control energy utilized along with the lowest error in performance, \(\star\) corresponds to the minimum energy utilized and the worst performance achieved and \(\Box\) corresponds to a compromise between the energy utilized and performance commanded.
The □ corresponds only to a much lower actuation used compared to ◊ along with only a slightly higher performance. It also gives much better performance compared to ⋄ by using slight higher actuation. It is interesting to note that the pitch rate error is more populated than the control power. Also, for the hypersonic vehicle a 1° δ_e may be significant. Hence, this study will try to avoid points which has δ_e > 1. The extreme point on the pareto front on the Control MS Norm axis is one such point. Figure 10(a) and Figure 10(b) shows the time responses for ◊ and □. The pitch rate (q) for □ shows worse transient response compared to ◊. Also, Figure 11(a) and Figure 11(b) shows that the control is utilized more for □ than for ⋄. So in essence, □ is a trade-off between performance and control actuation used. Based on the design requirements, the optimal design can be chosen from this pareto front.
Figure 9. Pareto front with time analysis performed for 3 points

Figure 10. Time response for 2 points shown on the pareto front. \(--\ ---- \) corresponds to \(\diamond\), \(--\ ---- \) corresponds to \(\Box\)

IV. Conclusion

This paper investigates a control-oriented design approach along with the introduction of frequency-dependent weighting functions in the \(\mathcal{H}_\infty\)-control synthesis framework as design variables. The central issue of this approach is the focus on controller existence and the optimal design simply finds the open-loop dynamics for which a control exists that achieves the lowest closed-loop metrics for optimal performance. This methodology is implemented for vibration-attenuation of a coupled and complex dynamics of a hypersonic vehicle. The objective of this case-study is to find the operating range to design the thermal protection system along with the optimal weightings for which a controller exists that minimize vibrations on the vehicle with 8 design variables. A multi-objective constrained optimization problem is formulated to achieve this objective and surrogates-based optimization using the NSGA-II is used for the optimization. This case study also highlights the benefits of using multiple surrogates as compared to single surrogates. This proposed methodology can be extended to more design variables and uncertainties can be incorporated without loss of generality.
Figure 11. Time response for 2 points shown on the pareto front. — corresponds to □, − − − corresponds to ⋆

Appendix

Initially, optimization is performed using the NSGA algorithm and only the best among the two PRS fits for the objectives, \( e_p, c \) and the constraint, \( \gamma \) are used. This optimization process is repeated for a total of three iterations adding 100 points at every iteration. Figure 12 shows the the pareto front for the initial data set of 700 points and after the final iteration for a total of 1000 points. As can be seen, the pareto front goes in the negative region of the normalized control energy, implying that the optimizer is indeed able to get better objective values. Also, more points are populated at regions where lower control energy is utilized and where the performance is between 0.8 and 1.5.

Figure 12. Pareto plots for the original set and after the final iteration

Optimization is then performed using the best two surrogates giving the lowest \( PRESS_{RMS} \) values in Table 3 i.e KRG and PRS. At the first iteration, 90 points are added using KRG fit and 90 points are added using 2nd-order PRS fit. \( P, K \) and simulation results are evaluated for the 180 points and the optimization is repeated for a second iteration. Figure 13 shows the pareto plot using only PRS (○) after 3 iterations and for a total of 1000 points and combination of KRG and PRS (◇) after 2 iterations and a total of 1100 points. The objective of the optimization is to get as close as possible to the origin on both the axis i.e. try to get the lowest 2-norm in the objective space. The combination of KRG and PRS seems to be adding more
points closer to the origin. Evidently, it is beneficial to use multiple surrogates in optimization for faster convergence.

Figure 13. Pareto front just for PRS (◦) and combination of PRS and KRG (♦)

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