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GENERALIZED FREQUENCY DOMAIN STATE-SPACE MODELS FOR ANALYZING FLEXIBLE ROTATING SPACECRAFT

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The mathematical model for a flexible spacecraft that is rotating about a single axis rotation is described by coupled rigid and flexible body degrees-of-freedom, where the equations of motion are modeled by integro-partial differential equations. Beam-like structures are often useful for analyzing boom-like flexible appendages. The equations of motion are analyzed by introducing generalized Fourier series for the deformational coordinate that transforms the governing equations into a system of ordinary differential equations. Though technically straightforward, two problems arise with this approach: (1) the model is frequency-truncated because a finite number of series terms are retained in the model, and (2) computationally intense matrix-valued transfer function calculations are required for understanding the frequency domain behavior of the system. Both of these problems are resolved by: (1) computing the Laplace transform of the governing integro-partial differential equation of motion; and (2) introducing a generalized state space (consisting of the deformational coordinate and three spatial partial derivatives, as well as single and double spatial integrals of the deformational coordinate). The Laplace domain equation of motion is cast in the form of a linear state-space differential equation that is solved in terms of a matrix exponential and convolution integral. The structural boundary conditions defined by Hamilton's principle are enforced on the closed-form solution for the generalized state space. The generalized state space model is then manipulated to provide analytic scalar transfer function models for the original integro-partial differential system dynamics. Symbolic methods are used to obtain closed-form eigen decomposition–based solutions for the matrix exponential/convolution integral algorithm. Numerical results are presented that compare the classical series based approach with the generalized state space approach for computing representative spacecraft transfer function models.

INTRODUCTION

Simplified models of maneuvering flexible spacecraft are often modeled as coupled rigid hub/beam-like structures. Mathematically, these systems are described by coupled systems of integro-partial differential equations (*IPDE*) (Turner, et. al, 1990, 1991, 1992; Lee and Junkins, 1992; Junkins and Kim, 1993). Classically generalized Fourier series models are introduced to transform the *IPDE* models into ordinary differential equations. This approach has been very successful. Nevertheless, high-accuracy solution strategies require many series terms, leading to

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large matrix equations for evaluating the system transfer functions. The need for dealing with large matrix equations is eliminated by introducing a generalized state-space (*GSS*) model that retains an exact s-domain flexible body model for the coupled rigid/flexible body system. Three steps are required for developing a transfer function model for the governing system *IPDE*: (1) the *IPDE* is Laplace transformed to yield a spatial *IPDE*; (2) the integral part of the *IPDE* is simplified by introducing integration-by-parts to yield a generalized integral equation involving multiple integrals of the beam variable; and (3) a *GSS* model is defined that replaces the *IPDE* model with a spatial 6x1 linear matrix-vector differential equation. No frequency truncation is introduced. A closed-form *s*-domain solution is obtained for the system matrix exponential and convolution integral by invoking symbolic methods. The coupled structural behavior for the vehicle is enforced by imposing the hub-beam boundary conditions obtained from an application of Hamilton's principle. Four boundary conditions are imposed on the structural response: (1) geometric boundary conditions for attaching the beam to the rigid hub (i.e., $y(r) = 0$, $y'(r) = 0$), and (2) satisfaction of the physical boundary conditions for the free end of the beam (i.e., $EI\left.\frac{\partial^2 y}{\partial x^2}\right|_{x=r+L} = 0$, $EI\left.\frac{\partial^3 y}{\partial x^3}\right|_{x=r+L} = 0$). With the geometric and physical boundary conditions satisfied, scalar transfer function models are obtained from the *GSS* model. Numerical results are presented to compare the accuracy and efficiency of both classical and the proposed *GSS* transfer function algorithms.

The major innovation of the paper is the introduction of a Laplace transformed-based *GSS* model for analyzing the behavior of an *IPDE* system dynamics model. The *GSS* model consists of the state, several partial derivatives of the state, as well as single and double integrals of the flexible body state variable (i.e., an *IPDE* state space). *GSS* is unconventional because it mixes variables evaluated at points in the structural domain, as well as integrals of the structural response over the entire flexible body domain. No Generalized Fourier Series representations are introduced for modeling the deformational *DOF*. Closed-form solutions are obtained for all state vector elements. The main contributions of this paper are: (1) the development of a closed-form Laplace transform-based solution for *6x6* matrix exponential that governs the behavior of a coupled hub-beam system, (2) rigorous scalar-valued distributed parameter transfer function models that are suitable for conducting engineering design iterations, and (3) the *GSS* model has no frequency truncation (it is effectively a closed-form solution for the coupled rigid hub/flexible body behavior).

Paper Organization

The paper is organized in the following way. The governing equation of motion (*EOM*) is presented in the math modeling section. The next section presents a classical transform function approach that transforms the *IPDE* model by introducing a generalized Fourier Series-based approach for the rotating rigid hub/beam dynamics problem. The *IPDE* model is transformed into a generalized integral equation by introducing an integration-by-parts for the flexible body angular momentum coupling integral. This transformation allows a linear spatial state space model to be developed for the *GSS* method. The next section develops a closed-form matrix exponential and convolution integral representation for the GSS state space model. The *s*-domain *6x6* matrix exponential is solved by introducing a mixed eigenvalue solution/symbolic algorithm. Three steps are required for manipulating the matrix exponential. First, the 4x4 beam sub-matrix of the matrix exponential is symbolically solved in closed-form by using a right- and left-eigenvalue solution strategy. Second, symbolic Taylor series methods are used to extend the 4x4 sub-problem to the complete 6x6 matrix exponential solution. Third, a complex-valued change of variables is introduced into the matrix exponential and convolution integral that replaces each element of the matrix exponential with a single complex-valued sine or cosine expression. The convolution matrix integral is solved in closed-form by introducing the transformed matrix exponential. The flexible body response is completely specified by imposing the physical beam boundary conditions at the free end of the beam, which yields the moment and shear boundary conditions at the hub/beam interface, yielding a complete mathematical model for the rigid hub/flexible beam system. The next section presents scalar-valued *GSS*-based transfer function models. The numerical results section presents several examples of both classical series-based approaches and the corresponding *GSS*-based transfer function results. The impacts of model truncation are examined in this section. The results of the paper are summarized in the conclusion section of the paper.

MATH MODEL

The coupled hub-beam model (see Fig. 1) combines a rotating rigid hub and cantilevered boom as a single subsystem. Only single-axis maneuvers are considered. The rigid hub is assumed to rotate about its local z-axis and the attached boom is allowed to have transverse deformations about the local y-axis. The undeformed boom lies along the local x-axis, where the attached appendage is assumed to be a uniform flexible beam subject to standard Euler-Bernoulli assumptions of negligible shear deformation and negligible distributed rotatory inertia. The *IPDE* is linearized by dropping the velocity component $-y\dot{\theta}$ term. Each substructure is modeled in terms of its kinetic and potential energy. The equations of motion (*EOM*) are developed by invoking Hamilton's extended principle (Meirovitch, 1991), yielding a coupled system of *IPDEs*, as well as the required geometric and physical boundary conditions for the beam structure.

The *IPDE* system is Laplace transformed and the resulting equations are manipulated into the form of a linear matrix differential equation; yielding a closed-form s-domain spatial matrix exponential/convolution integral solution. A two-part derivation approach is presented for the s-domain *6x6* matrix exponential: (1) The *4x4* beam sub-part of the matrix is solved in closed-form using a symbolic eigenvalue solution method; and (2) The *4x4* beam closed-form solution is used to recover closed-form solutions for the remaining elements of the coupled *6x6* hub/beam *GSS* matrix exponential elements. The symbolically obtained solutions are validated by comparing series-based solutions with Taylor expansions of the closed-form solutions. A symbolic bi-orthogonal eigen decomposition algorithm is applied for the *4x4* beam sub-matrix part of the matrix exponential (Junkins and Turner, 1985) that provides an analytic solution. The full matrix exponential is recovered by symbolically Taylor expanding the matrix and comparing results with the beam sub-matrix at the element-by-element level. The convolution matrix integral is evaluated in closed-form.

Future implementations of these substructure models are expected to produce significant computational reductions in the computational effort required to analyze the frequency domain behaviors for engineering-level-of-fidelity models for rotating rigid bodies with attached beam-like structures.

Kinetic and Potential Energy

The kinetic and potential energies of the coupled rigid hub/beam hybrid system are given by (Turner, et. al., 1991; Lee and Junkins 1992; Junkins and Kim, 1993):

$$2T = J_h\dot{\theta}^2 + \int_r^{r+l}\left[\rho\left(\dot{y}+x\dot{\theta}\right)^2\right]dx; \qquad 2V = \int_r^{r+l}\left[EI\left(y_{,xx}\right)^2\right]dx$$

The nonconservative virtual work for this system follows as

$$\delta W_{nc} = u\delta\theta$$

where **θ** denotes the rigid body rotation angle for central rigid hub, ***y*** denotes the transverse beam deflection coordinate, ***E*** denotes the elastic modulus of the beam, J denotes the moment of combined inertia for the rigid hub and the undeformed beam, I denotes the moment of inertia for the beam, ρ denotes the linear beam mass density, A denotes the beam cross sectional area, and u denotes the torque applied to the rigid hub.

Rotating linked Hub and Flexing Beam Math Model

Application of Hamilton's principle (Meirovitch,1991) for the model presented in Figure 2 leads to the *EOM* and governing geometric/physical boundary conditions:

Integro-Partial Differential Equation of Motion Model (IPDE). The math model for the vehicle consists of coupled angular momentum and elastic beam equations, as follows.

$$J\ddot{\theta} + \int_r^{r+l} \underbrace{\rho A x \ddot{y} dx}_{\text{Bending Ietegral Coupling}} = u \qquad J = \underbrace{J_h}_{\text{Hub Inertia}} + \underbrace{\int_r^{r+l} \rho A x^2 dx}_{\text{Undeformed Beam Inertia}} \tag{1}$$

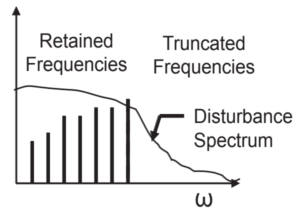
$$\rho A\left(\ddot{y} + \underbrace{x\ddot{\theta}}_{\text{Rigid Body Coupling}}\right) + EIy_{,xxxx} = 0 \tag{2}$$

where the explicit x-axis dependence complicates the subsequent analysis. Hamilton's principle provides the geometric and physical boundary conditions listed in Table 1.

Table 1: Geometric & Physical Boundary Conditions

Boundary Conditions	Model
Displacement	$y\big\|_r = 0$
Slope	$y_{,x}\big\|_r = 0$
Bending Moment	$EI\dfrac{\partial^2 y}{\partial x^2}\Big\|_{r+l} = 0$
Shear Force	$EI\dfrac{\partial^3 y}{\partial x^3}\Big\|_{r+l} = 0$

Equations (1) and (2) are hard to solve because both time and space variables appear. Classically this problem is handled by introducing a generalized Fourier series, where displacement shapes are provided for eliminating the spatial dependence after integration over the spatial domain. A linear matrix-vector system of ordinary differential equations is obtained for the coupled hub/flexible beam system. The model is frequency truncated because only a finite number of terms are retained in the generalized Fourier series representation. Model truncation carries with it the risk that disturbance sources with significant energy in the range of the truncated frequencies can lead to poor structural response predictions. Model truncation is eliminated in this paper by combining the Laplace transform method and *GSS*.



Classical Series Expansion Approach for Modeling Hub-Beam Structures. A brief review of the classical transfer function approach follows to facilitate the comparison of classical and GSS-based Transfer function methods. The structural response is analyzed by assuming a Generalized Fourier series (Meirovitch, 1967, 1991):

$$y(x,t) = \sum_{r=1}^{N} \phi_r(x)\eta_r(t) = \underbrace{\Phi^T(x)}_{1xN}\underbrace{\Upsilon(t)}_{Nx1}$$

where $\phi_r(x)$ denotes the r-th flexural displacement shape and $\eta_t(t)$ denotes the modal amplitude for the flexural displacement shape. Assuming the power series model, Eqs. (1) and (2) become

$$J\ddot{\theta} + \left(\int_r^{r+l} \rho A x \Phi^T(x)dx\right)\ddot{\Upsilon} = u$$

$$\rho A \Phi^T(x)\ddot{\Upsilon} + \rho A x \ddot{\theta} + EI\Phi^T_{,xxxx}(x)\Upsilon = 0$$

The spatial dependence is eliminated from the second equation by pre-multiplying by the displacement vector and integrating over the domain of interest, leading to the transformed equations

$$J\ddot{\theta} + \left(\int_r^{r+l} \rho A x \Phi^T(x)dx\right)\ddot{\Upsilon} = u$$

$$\left(\int_r^{r+l} \rho A \Phi(x)\Phi^T(x)dx\right)\ddot{\Upsilon} + \left(\int_r^{r+l} \rho A x \Phi(x)\,dx\right)\ddot{\theta} + \left(EI\int_r^{r+l} \Phi(x)\Phi^T_{,xxxx}(x)dx\right)\Upsilon = 0$$

or

$$M\ddot{x} + Kx = f \tag{3}$$

where

$$M = \begin{bmatrix} J & M_\theta^T \\ M_\theta & M_{\phi\phi} \end{bmatrix} \qquad K = \begin{bmatrix} 0 & 0^T \\ 0 & K_{\phi\phi} \end{bmatrix} \qquad M_\theta = \int_r^{r+l} \rho A x \Phi(x)\,dx$$

$$M_{\phi\phi} = \int_r^{r+l} \rho A \Phi(x)\Phi^T(x)dx \qquad K_{\phi\phi} = EI\int_r^{r+l} \Phi(x)\Phi^T_{,xxxx}(x)dx$$

Laplace Transform Model. The integral definition of the Laplace transform is given by

$$\overline{f}(\mu) = \int_0^\infty e^{-\mu t} f(t) dt$$

Applied to Eq. (3) one obtains the matrix-vector transfer function for the coupled hub/beam problem.

$$\mu^2 M\overline{x} + K\overline{x} = \overline{f} \qquad \overline{x} = \left[\mu^2 M + K\right]^{-1} \overline{f} \tag{4}$$

The computational cost associated with evaluating the transfer function is dominated by the symmetric matrix inversion calculation which scales as n^3, where n denotes the number of modes retained in the model. A symmetric matrix inversion is computed for each value of frequency. Alternatively, the *IPDE* transfer function calculation presented in this paper replaces the expensive frequency truncated matrix inversion calculation with a scalar inversion calculation where an exact frequency domain model is retained.

The Laplace Transform of the *EOM* defined by Eqs. (1) and (2), yields the coupled pair of *spatial IPDE*'s

$$\begin{gathered}\mu^2 J\overline{\theta} + \mu^2 \rho A \int_r^{r+l} x\overline{y}\,dx = \overline{u} \\ EI\,\overline{y}_{,xxxx} + \mu^2 \rho A\left(\overline{y} + x\overline{\theta}\right) = 0\end{gathered} \tag{5}$$

The development of a *GSS* model is simplified by replacing the integral term with the equivalent *integration-by-parts* solution

$$\int_r^{r+l} x\overline{y}dx = x\int_r^{r+l} \overline{y}dx - \iint_r \overline{y}dxdx'$$

which replaces the integrations with integrands only involving the Laplace transform of the flexible body distributed parameter variable. Introducing the integration-by-parts solution into Eq. (5) yields the desired form for the *spatial IPDE EOM*

$$\begin{gathered}\mu^2 J\overline{\theta} + \mu^2 \rho A\left(x\int_r^{r+l} \overline{y}dx - \iint_r \overline{y}dxdx'\right) = \overline{u} \\ EI\,\overline{y}_{,xxxx} + \mu^2 \rho A\left(\overline{y} + x\overline{\theta}\right) = 0\end{gathered} \tag{6}$$

which is now classified as a *generalized integral equation* because of the appearance of the double integral. The second equation is an ordinary 4th order differential equation. Both equations must be solved simultaneously. A standard state space approach is not equal to the task: a new state space is required that simultaneously handles all of the flexible body terms, including: $\overline{y}_{xxxx}$, $\overline{y}$, $\int \overline{y}dx$, and $\int\int \overline{y}dxdx'$.

Spatial Integro-Differential State Space Model

Four partial derivatives and two integrals must be analyzed for generating the flexible body variable $\overline{y}(x)$. $\overline{\theta}$ is not included in the *GSS* because it has no spatial dependence. To this end, defining the spatial derivative as $(*)' = \partial(*)/\partial x$, the state variables for the *GSS* follow as

$$
\begin{aligned}
&z_1 = \iint \bar{y}\,dx\,dx' && z_1' = z_2 \\
&z_2 = \int \bar{y}\,dx && z_2' = z_3 \\
&z_3 = \bar{y} && z_3' = z_4 \\
&z_4 = \bar{y}' && z_4' = z_5 \\
&z_5 = \bar{y}'' && z_5' = z_6 \\
&z_6 = \bar{y}''' && z_6' = -\beta\left(z_3 + x\bar{\theta}\right) \qquad \beta = \frac{\mu^2 \rho A}{EI}
\end{aligned}
\tag{7}
$$

The state space is generalized in the sense that both partial derivatives and integral terms appear explicitly in the variable definitions. The rigid body motion appears as a forcing function. All variables appear linearly, allowing closed-form solutions to be obtained for the matrix exponential-based algorithm.

Boundary-Value Solution for the GSS Initial Conditions. The *GSS* initial conditions are given by

$$
Z\big|_{x=0} = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 & z_6 \end{bmatrix}^t \Big|_{x=0}
$$

The first two terms are zero because the integrals are assumed to be zero initially. The third and fourth terms vanish because of the geometrical boundary conditions defined by Table 1, leading to

$$
Z\big|_{x=0} = \begin{bmatrix} 0 & 0 & 0 & 0 & z_5 & z_6 \end{bmatrix}^t \Big|_{x=0} \tag{8}
$$

The remaining solutions for z_5 and z_6 are defined by enforcing the physical boundary conditions defining force and moment balance conditions at the end of the beam (see Table 1). These boundary conditions are evaluated after the complete spatial *IPDE* solution has been analytically integrated.

Using the variable definitions in Eq. (7) the rigid body equation of Eq. (6) is expressed as

$$
\mu^2 J\bar{\theta} + \mu^2 \rho A\left(x\,z_2 - z_1\right) = \bar{u} \tag{9}
$$

where the second term on the left provides an exact representation for the distributed parameter beam coupling effects for the rigid-hub/beam system.

μ-Domain Spatial Linear Matrix Differential Equation. Equation (7) is re-cast as the following *6x6* linear matrix differential equation

$$Z' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\beta & 0 & 0 & 0 \end{bmatrix} Z + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -\beta x\bar{\theta} \end{Bmatrix} = [A]Z + b \tag{10}$$

which has the well-known solution (Junkins and Turner, 1985; Junkins and Kim, 1993)

$$Z\big|_{x=l} = \exp[Al]Z\big|_{x=0} + \int \exp[A(l-\tau)]b(\tau)\,d\tau \tag{11}$$

The matrix exponential of Eq. (11) is easily evaluated by developing a series expansion.

$$\exp[Al] = I + Al + \frac{1}{2}(Al)^2 + \frac{1}{6}(Al)^3 + \cdots$$

Nevertheless, when many values of μ must be evaluated, computational efficiency becomes a serious issue. To reduce the computational impact of repeated matrix evaluations, we seek a closed-form solution for the matrix exponential. To this end, a two-step approach is presented: (1) the beam 4x4 sub-problem is solved using a symbolic eigen decomposition technique, and (2) the 6x6 *GSS* solution is obtained using analytical insights gained from the 4x4 beam sub-problem solution. This strategy decouples the flexible body part of the solution from the extended calculation involving first and second integrals of the flexible body response.

Closed-Form Solution for the Spatial State Matrix Exponential. The Laplace transform for the beam part of the EOM is defined as

$$EI\,\bar{y}_{,xxxx} + \mu^2 \rho A\left(\bar{y} + x\bar{\theta}\right) = 0$$

where rigid body coupling term $\mu^2 \rho A x\bar{\theta}$ acts as a forcing term. In terms of the *GSS* defined by Eq. (7), the unforced beam sub-problem consists of the *4x4* beam bending linear matrix differential equation given by

$$Q' = CQ \qquad Q = \exp[Cx]Q\big|_{x=0} \tag{12}$$

where

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & 0 & 0 \end{bmatrix} \tag{13}$$

The solution for Eq. (12) is obtained from the eigen decomposition of *C* in terms of its right- and left-eigenvectors. The eigen decomposition for *C* is obtained symbolically by using the computer aided algebra program MACSYMA. The resulting analytic solution is shown to exhibit group-like properties for the elements of the solution. The closed-form solutions obtained for Eq. (12) are validated by Taylor expanding the analytic solutions and comparing the results with the series expansions for the matrix exponentials. The insights gleaned from solving Eq. (12) guide the solution strategy for obtaining a closed-form solution for the *6x6* system appearing in Eq. (10).

A. Beam Sub-Problem Matrix Exponential_A closed-form solution is obtained for $Q' = CQ$ by exploiting the bi-orthogonality conditions for the eigenvectors for C (Junkins and Turner, 1985; Junkins and Kim, 1993). The *right*- and *left-eigenvectors* are obtained by solving the eigenvalue problems:

$$\begin{aligned} CR_i &= \lambda_i R_i, \quad i \quad = 1, 2, \cdots, n \\ C^t L_j &= \lambda_j L_j, \quad i \quad = 1, 2, \cdots, n \\ L_j^t R_i &= \delta_{ij}, \quad i, j = 1, 2, \cdots, n \end{aligned} \tag{14}$$

The *4x4* diagonal matrix containing the eigenvalues for *C* is given by

$$D = \beta^{1/4}\,\text{Diag}\begin{bmatrix} -\dfrac{1}{\sqrt{i}} & -\sqrt{i} & \dfrac{1}{\sqrt{i}} & \sqrt{i} \end{bmatrix}$$

where $i = \sqrt{-1}$ denotes an imaginary complex number. With *R*, *L* and *D* known, one can express the *C* matrix in terms of its eigen decomposition. Observe that the eigenvectors are normalized by $L^T R = I \Rightarrow R^{-1} = L^T$. From Eq. (14) it follows that one can write $CR = RD$. Post multiplying this result by R^{-1} leads to desired result

$$C = RDR^{-1} = RDL^T$$

Introducing *C* into the matrix exponential and factoring out the *right*- and *left-eigenvectors*, yields

$$\exp\left[Cx\right] = \exp\left[RDL^T x\right] = \mathrm{R}\exp[Dx]L^T \tag{15}$$

where

$$\exp[Dx] = \begin{bmatrix} e^{-\beta^{1/4}x/\sqrt{i}} & 0 & 0 & 0 \\ 0 & e^{-\sqrt{i}\beta^{1/4}x} & 0 & 0 \\ 0 & 0 & e^{\beta^{1/4}x/\sqrt{i}} & 0 \\ 0 & 0 & 0 & e^{\sqrt{i}\beta^{1/4}x} \end{bmatrix}$$

The computer aided algebra program MACSYMA 2.4 has been used to solve Eq. (15) . The final results for the *right*- and *left-eigenvectors* are not presented because of space limitations. A careful examination of the symbolic results provided for Eq. (15) indicates that the matrix exponential encodes the following repetitive sub-structure:

$$\exp\left[Cx\right] = \begin{bmatrix} f & -f'''/\beta & -f''/\beta & -f'/\beta \\ f' & f & -f'''/\beta & -f''/\beta \\ f'' & f' & f & -f'''/\beta \\ f''' & f'' & f' & f \end{bmatrix} \tag{16}$$

where the following matrix element exists along the diagonal

$$f\left(x\right) = \cos\left(\frac{\beta^{1/4}x}{\sqrt{2}}\right)\cosh\left(\frac{\beta^{1/4}x}{\sqrt{2}}\right) \tag{17}$$

The identification of the matrix structure in Eq. (16) in terms of the derivatives of *f* provides the critical insight required for completely solving the *6x6 GSS* matrix exponential of Eq. (10). Equations (15) and (16) are checked by symbolically evaluating the following expansion

$$I + Cx + C^2x^2/2! + \cdots + C^N x^N / N! \approx \exp\left[Cx\right]$$

B. GSS Matrix Exponential Solution Comparing Taylor expansions for $\exp\left[Ax\right]$ and $\exp\left[Cx\right]$, one easily establishes the following data structure for $\exp\left[Ax\right]$:

$$\exp\left[Ax\right] = \begin{bmatrix} 1 & x & -f''/\beta & -f'/\beta & \left(1-f\right)/\beta & \left(\beta x + f'''\right)/\beta^2 \\ 0 & 1 & -f'''/\beta & -f''/\beta & -f'/\beta & \left(1-f\right)/\beta \\ 0 & 0 & f & -f'''/\beta & -f''/\beta & -f'/\beta \\ 0 & 0 & f' & f & -f'''/\beta & -f''/\beta \\ 0 & 0 & f'' & f' & f & -f'''/\beta \\ 0 & 0 & f''' & f'' & f' & f \end{bmatrix} \tag{18}$$

where the 4x4 beam sub-structure matrix exponential is preserved.

C. Inverse Matrix Exponential Solution The inverse of the matrix exponential appearing in Eq. (18) is required for the forced convolution part of the closed-form solution provided by Eq. (11). By a similar process one can establish that the inverse matrix exponential is given by

$$\exp\left[-Ax\right] = \begin{bmatrix} 1 & -x & -f''/\beta & f'/\beta & \left(1-f\right)/\beta & -\left(\beta x + f'''\right)/\beta^2 \\ 0 & 1 & f'''/\beta & -f''/\beta & f'/\beta & \left(1-f\right)/\beta \\ 0 & 0 & f & f'''/\beta & -f''/\beta & f'/\beta \\ 0 & 0 & -f' & f & f'''/\beta & -f''/\beta \\ 0 & 0 & f'' & -f' & f & f'''/\beta \\ 0 & 0 & -f''' & f'' & -f' & f \end{bmatrix} \tag{19}$$

This equation has been checked by introducing Eq. (17) into Eqs. (18) and (19), multiplying the results, and applying trig identities to confirm that the product is a *6x6* identity matrix.

D. Complex Form of GSS Matrix Exponential A very compact form of Eqs. (18) and (19) is obtained by recognizing that *f* in Eq. (17) represents the *real part* of the following complex function.

$$f = \cos\left(\boldsymbol{\sigma} x\right) \tag{20}$$

where

$$\sigma = \beta^{1/4}\left(1+i\right)/\sqrt{2} = \sqrt{i\sqrt{\beta}} \tag{21}$$

To this end, one can express the matrix exponential function of Eq. (18) in the transformed compact form

$$\exp\left[Ax\right]=\begin{bmatrix} 1 & x & \boldsymbol{\sigma}^2 c/\beta & \boldsymbol{\sigma} s/\beta & \left(1-c\right)/\beta & \left(\boldsymbol{\sigma}^3 s+\beta x\right)/\beta^2 \\ 0 & 1 & -\boldsymbol{\sigma}^3 s/\beta & \boldsymbol{\sigma}^2 c/\beta & \boldsymbol{\sigma} s/\beta & \left(1-c\right)/\beta \\ 0 & 0 & c & -\boldsymbol{\sigma}^3 s/\beta & \boldsymbol{\sigma}^2 c/\beta & \boldsymbol{\sigma} s/\beta \\ 0 & 0 & -\boldsymbol{\sigma} s & c & -\boldsymbol{\sigma}^3 s/\beta & \boldsymbol{\sigma}^2 c/\beta \\ 0 & 0 & -\boldsymbol{\sigma}^2 c & -\boldsymbol{\sigma} s & c & -\boldsymbol{\sigma}^3 s/\beta \\ 0 & 0 & \boldsymbol{\sigma}^3 s & -\boldsymbol{\sigma}^2 c & -\boldsymbol{\sigma} s & c \end{bmatrix} \tag{22}$$

where $s=\sin\left(\boldsymbol{\sigma}x\right)$ and $c=\cos\left(\boldsymbol{\sigma}x\right)$. It is indeed remarkable that the coupled rigid hub-flexible beam *IPDE* is mathematically described by such a simple matrix exponential. Equation (22) has been validated by (1) carrying out Taylor expansions of Eq. (22), (2) collecting the real part, and (3) expanding exp[Ax] to the same order, and comparing individual terms. Not surprisingly this simple change of variables also greatly simplifies the convolution integral calculations for the forced part of the solution for Eq. (10).

Closed-Form Solution Components. The solution for GSS is completed by introducing Eq. (22) into

$$Z\left(x\right)=\exp\left[Ax\right]Z_0+\int_0^x \exp\left[A\left(x-\tau\right)\right]b\left(\tau\right)d\tau \tag{23}$$

and recalling the initial condition vector defined by Eq. (8). The *homogenous term* is straightforward and can be shown to be

$$Z_H(x)=\begin{pmatrix} (1-c(\boldsymbol{\sigma}x)z_5/\beta+\left(\boldsymbol{\sigma}^3 s(\boldsymbol{\sigma}x)+\beta x\right)z_6/\beta^2 \\ \boldsymbol{\sigma}s(\boldsymbol{\sigma}x)z_5/\beta+\left(1-c(\boldsymbol{\sigma}x)\right)z_6/\beta \\ \boldsymbol{\sigma}^2 c(\boldsymbol{\sigma}x)z_5/\beta+\boldsymbol{\sigma}s(\boldsymbol{\sigma}x)z_6/\beta \\ -\boldsymbol{\sigma}^3 s(\boldsymbol{\sigma}x)z_5/\beta+\boldsymbol{\sigma}^2 c(\boldsymbol{\sigma}x)z_6/\beta \\ c(\boldsymbol{\sigma}x)z_5-\boldsymbol{\sigma}^3 s(\boldsymbol{\sigma}x)z_6/\beta \\ -\boldsymbol{\sigma}s(\boldsymbol{\sigma}x)z_5+c(\boldsymbol{\sigma}x)z_6 \end{pmatrix} \tag{24}$$

The *forced solution* second term becomes

$$\int_0^x \exp\left[A\left(x-\tau\right)\right]b\left(\tau\right)d\tau=\int_0^x \exp\left[A\left(x-\tau\right)\right]\begin{pmatrix}0 & 0 & 0 & 0 & 0 & -\beta\tau\bar{\theta}\end{pmatrix}^T d\tau$$

which reduces to

$$\int_0^x \exp\left[A\left(x-\tau\right)\right]b\left(\tau\right)d\tau=-\beta\bar{\theta}\int_0^x \tau\begin{pmatrix} \left(\beta\left(x-\tau\right)+f'''\left(x-\tau\right)\right)/\beta^2 \\ \left(1-f\left(x-\tau\right)\right)/\beta \\ -f'\left(x-\tau\right)/\beta \\ -f''\left(x-\tau\right)/\beta \\ -f'''\left(x-\tau\right)/\beta \\ f\left(x-\tau\right) \end{pmatrix}d\tau \tag{25}$$

Introducing the complex variable form for *f* defined by Eq. (20), and integrating the vector set of terms defined by Eq. (25), leads to

$$\int_0^x \exp\left[A(x-\tau)\right]b(\tau)\,d\tau = \begin{pmatrix} I_1 & I_2 & I_3 & I_4 & I_5 & I_6 \end{pmatrix}^T$$

where the individual convolution integrals can be shown to be

$$I_1 = -\frac{\bar{\theta}\left(-\boldsymbol{\sigma}\sin(\boldsymbol{\sigma}x) + \frac{\beta x^3}{6} + \boldsymbol{\sigma}^2 x\right)}{\beta} \quad I_2 = -\frac{\bar{\theta}}{\boldsymbol{\sigma}^2}\left(\cos(\boldsymbol{\sigma}x) - 1 + \frac{\boldsymbol{\sigma}^2 x^2}{2}\right) \quad I_3 = -\boldsymbol{\sigma}\bar{\theta}\left(\frac{x}{\boldsymbol{\sigma}} - \frac{\sin(\boldsymbol{\sigma}x)}{\boldsymbol{\sigma}^2}\right) \tag{26}$$

$$I_4 = -\bar{\theta}\left(1-\cos(\sigma x)\right) \quad I_5 = \bar{\theta}\left(\boldsymbol{\sigma}^2 x - \boldsymbol{\sigma}\sin(\boldsymbol{\sigma}x)\right) \quad I_6 = \frac{\beta\bar{\theta}}{\boldsymbol{\sigma}^2}\left(1-\cos(\boldsymbol{\sigma}x)\right)$$

A. Terminal Physical Boundary Conditions for the Beam Bending Solution The homogeneous and forced parts of the spatial solution are collected as

$$Z(x) = \begin{pmatrix} (1-c(\boldsymbol{\sigma}x)z_5/\beta + \left(\boldsymbol{\sigma}^3 s(\boldsymbol{\sigma}x) + \beta x\right)z_6/\beta^2 + I_1(x) \\ \boldsymbol{\sigma}s(\boldsymbol{\sigma}x)z_5/\beta + \left(1-c(\boldsymbol{\sigma}x)\right)z_6/\beta + I_2(x) \\ \boldsymbol{\sigma}^2 c(\boldsymbol{\sigma}x)z_5/\beta + \boldsymbol{\sigma}s(\boldsymbol{\sigma}x)z_6/\beta + I_3(x) \\ -\boldsymbol{\sigma}^3 s(\boldsymbol{\sigma}x)z_5/\beta + \boldsymbol{\sigma}^2 c(\boldsymbol{\sigma}x)z_6/\beta + I_4(x) \\ c(\boldsymbol{\sigma}x)z_5 - \boldsymbol{\sigma}^3 s(\boldsymbol{\sigma}x)z_6/\beta + I_5(x) \\ -\boldsymbol{\sigma}s(\boldsymbol{\sigma}x)z_5 + c(\boldsymbol{\sigma}x)z_6 + I_6(x) \end{pmatrix} \tag{27}$$

The only unknowns are the boundary conditions for z_5 and z_6; which are recovered by enforcing the moment and shear conditions at the end of the beam. The last two equations of Eq. (27) have the information required for solving for the unknowns, leading to the necessary condition for the physical boundary conditions given by

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{bmatrix} c(\boldsymbol{\sigma}L) & -\boldsymbol{\sigma}^3 s(\boldsymbol{\sigma}L)/\beta \\ -\boldsymbol{\sigma}s(\boldsymbol{\sigma}L) & c(\boldsymbol{\sigma L}) \end{bmatrix}\begin{pmatrix} z_5 \\ z_6 \end{pmatrix} + \begin{pmatrix} I_5(L) \\ I_6(L) \end{pmatrix}$$

which is analytically inverted for the initial condition parameters, yielding

$$\begin{pmatrix} z_5 \\ z_6 \end{pmatrix} = \frac{\beta\bar{\theta}}{c(\boldsymbol{\sigma L})^2 - \boldsymbol{\sigma}^4 s(\boldsymbol{\sigma L})^2/\beta}\begin{bmatrix} c(\boldsymbol{\sigma L}) & \boldsymbol{\sigma}^3 s(\boldsymbol{\sigma L})/\beta \\ \boldsymbol{\sigma}s(\boldsymbol{\sigma L}) & c(\boldsymbol{\sigma L}) \end{bmatrix}\begin{pmatrix} I_5(L) \\ I_6(L) \end{pmatrix} \tag{28}$$

Introducing the convolution integrals defined by Eq. (26), completes the solution for the initial coefficients, leading to

$$z_5 = -\frac{\beta\sigma\left(\mathrm{s}(\sigma L) - \sigma L\,\mathrm{c}(\sigma L)\right)\bar{\theta}}{\sigma^4\,\mathrm{s}^2(\sigma L) - \beta\,\mathrm{c}^2(\sigma L)}$$

$$z_6 = -\frac{\beta\left(\sigma^4\,\mathrm{s}(\sigma L)\left(\mathrm{s}(\sigma L) - \sigma L\right) - \beta^2\,\mathrm{c}(\sigma L)\left(\mathrm{c}(\sigma L) + 1\right)\right)\bar{\theta}}{\sigma^2\left(\sigma^4\,\mathrm{s}^2(\sigma L) - \beta\,\mathrm{c}^2(\sigma L)\right)}$$

The full *GSS* solution process is completed by introducing these equations into Eq. (27), and substituting the complex parameter for σ defined by Eq. (21) into the resulting equation. The desired solution is recovered by evaluating the real part of the equations as

$$Y(x) = \Re e\left(Z(x)\right)\bar{\theta} \tag{29}$$

where $\bar{\theta}$ is factored out of the vector of solutions. This complicated step is preformed symbolically. Equation (29) provides the response solution required for the transfer function calculations.

Transfer Function Calculations

Transfer function calculations are presented for rotational and flexible body coupling effects.

Rotational Transfer Function. Introducing the initial condition solution provided by Eq. (24) into Eq. (8) and the result into Eq. (9) yields the single and double integral solutions

$$\begin{aligned}
Z_1 &= \left((1-c)z_5 + (\sigma^3 s + \beta x)z_6\right)\Big/\beta \\
&= \frac{\bar{\theta}}{c^2 - \sigma^4 s^2/\beta}\begin{pmatrix} 1-c \\ \sigma^3 s + \beta x \end{pmatrix}^T \begin{bmatrix} c & \sigma^3 s/\beta \\ \sigma s & c \end{bmatrix}\begin{pmatrix} I_5(L) \\ I_6(L) \end{pmatrix} \\
&= \bar{\theta}\, g_5(s) \\
Z_2 &= \frac{\bar{\theta}}{c^2 - \sigma^4 s^2/\beta}\begin{pmatrix} \sigma s \\ 1-c \end{pmatrix}^T \begin{bmatrix} c & \sigma^3 s/\beta \\ \sigma s & c \end{bmatrix}\begin{pmatrix} I_5(L) \\ I_6(L) \end{pmatrix} \\
&= \bar{\theta}\, g_6(s)
\end{aligned}$$

Introducing the integral terms in the angular momentum equation leads to

$$s^2\left(J + \rho A\left(xZ_2(x,s) - Z_1(x,s)\right)\right)\bar{\theta} = \bar{u} \tag{30}$$

which is inverted for the hub rotational transfer function given by

$$\bar{\theta} = \frac{\bar{u}}{s^2\left(J + \rho A\left(xg_5(x,s) - g_6(x,s)\right)\right)} \tag{31}$$

Flexible Body Transfer Function. The flexible body element is defined by the third element the GSS state as

$$
\begin{aligned}
\bar{y} &= \left(\sigma^2 c z_5 + \sigma s z_6\right) \big/ \beta \\
&= \frac{\bar{\theta}}{c^2 - \sigma^4 s^2/\beta} \begin{pmatrix} \sigma^2 c \\ \sigma s \end{pmatrix}^T \begin{bmatrix} c & \sigma^3 s/\beta \\ \sigma s & c \end{bmatrix} \begin{pmatrix} I_5(L) \\ I_6(L) \end{pmatrix} \\
&= \bar{\theta} g_3(s) \\
&= \frac{g_3(x,s)\bar{u}}{s^2\left(J + \rho A\left(x g_5(x,s) - g_6(x,s)\right)\right)}
\end{aligned}
\tag{32}
$$

It is obvious that Eqs. (31) and (32) are scalar equations and easily computed. This is in stark contrast to the situation where series approximations are used and the respective transfer functions require the numerical inversion of high-order matrix inversion algorithms.

NUMERICAL RESULTS

The hub/beam model parameters are presented in Table 1, which have been selected for demonstration purposes only; the parameters do not represent a physical structure and the numbers are assumed to be non-dimensional.

Parameter	Description	Value
m_h	Mass of the hub	16
m_a	Mass of the arm / beam	0.10875
D_h	Diameter of hub	2
L	Length of the arm / beam	1
E	Bending Stiffness of arm	1
I_h	Hub moment of inertia	1
ρ	Mass density of arm	1
A	Cross sectional area of arm	1
J	Total inertia	$J = I_h + \frac{\rho A L^3}{3} = 1.33$

The matrix-valued transfer function of Eq. (4) is evaluated by defining an assumed shape for the deformation behavior for the beam. Numerical results are reported for an 8-mode model where the assumed mode is defined by (Junkins and Turner, 1986)

$$\phi(x) = 1 - \cos\left(\frac{p\pi x}{L}\right) + \frac{1}{2}(-1)^{p+1}\frac{p\pi x^2}{L}; \qquad p = 1, 2, \cdots, 8 \tag{33}$$

Equation (33) satisfies the geometric boundary conditions at the attach point between the hub and the flexible beam. The moment boundary condition at the free end of the beam is not satisfied, however, the shear boundary condition is satisfied. The structural integrals are evaluated symbolically and the matrices are built and evaluated using Matlab. The GSS transfer functions are processed symbolically and numerically evaluated. Several plots are presented that depict the recovered transfer functions as a function of frequency for several points along the beam. The Figure 2 presents the transfer function evaluated at x = 0.1. In every case the assumed modes frequency estimates converge from above as the number of modes increase. As the frequency increases one can observe that the error in the frequency becomes more pronounced.

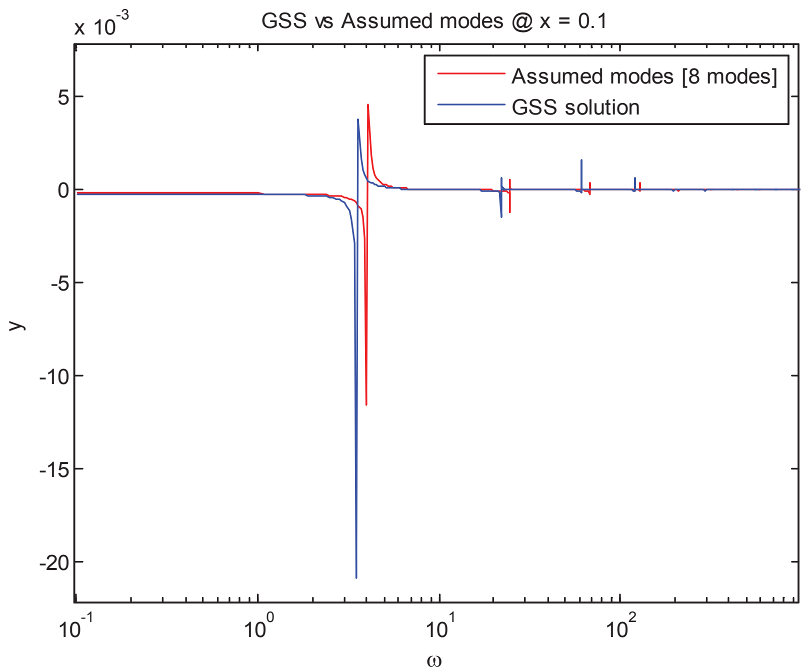


Figure 2: GSS and Classical Transfer Function Evaluated at x = 0.1

In Figures 3 and 4 as one moves further away from the hub the GSS model captures a greater structural response. The beam tip response is very small. Even though the 8 modes are used for the assumed modes method it is clear that significant errors exist in the transfer function response predictions, which is important to understand for control strategies and the sensitivities of these algorithms to plant errors in the frequency estimates.

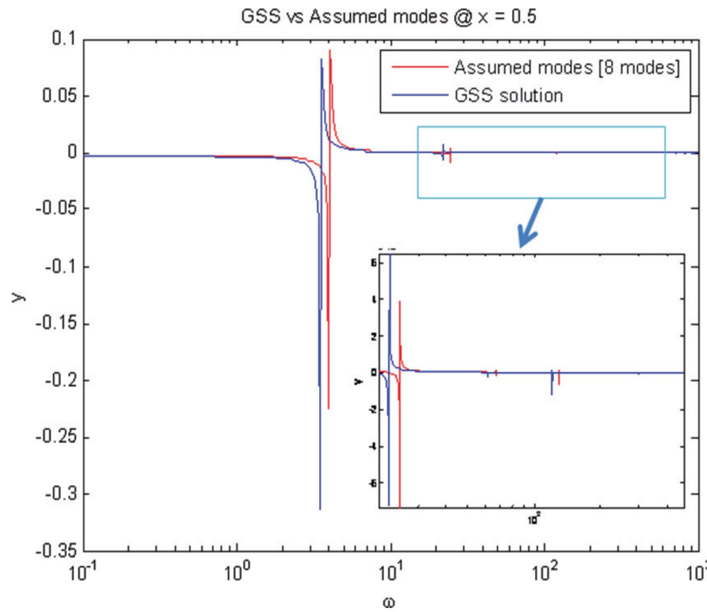


Figure 3: GSS and Classical Transfer Function Evaluated at x = 0.5

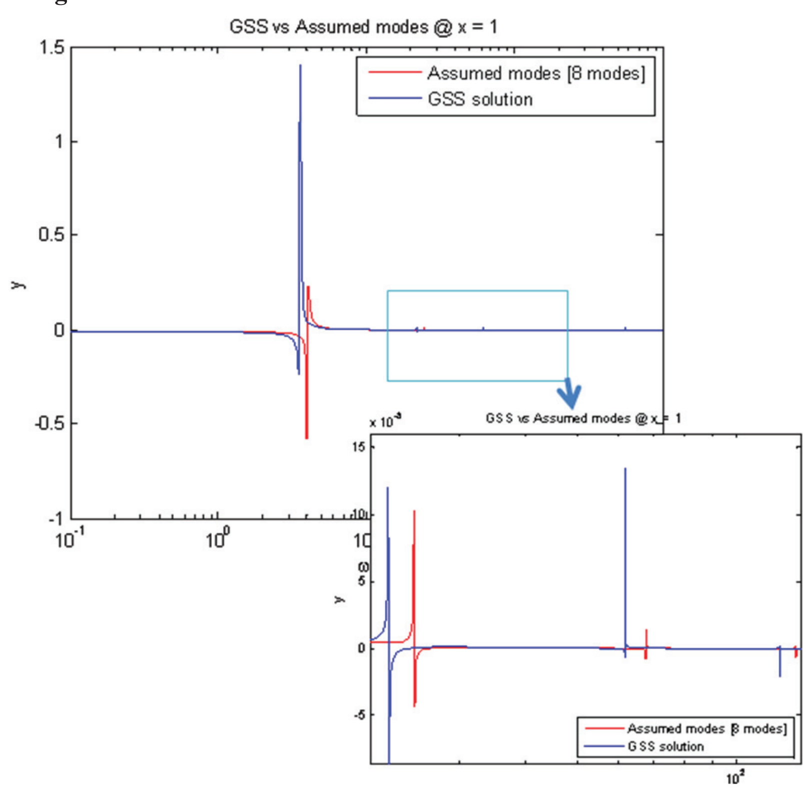


Figure 4: GSS and Classical Transfer Function Evaluated at x = 1.0 (Free End)

A second numerical experiment is presented, where the hub inertial is varied from zero to large values to sweep the range of beam behaviors from free-free boundary conditions to cantilever boundary conditions. The results of these experiments is summarized in **Table 2**, where the beam frequencies are seen to rapidly decrease as the hub inertia in increased from zero to a large value.

Hub Inertia	Frequencies (rad/s)				
0	15.4	50.0	104.2	178	272
0.005	14.3	37.6	69.5	125	202
0.01	13.3	31.5	65.3	123	201
1	4.01	22.2	61.7	121	200
8	3.50	22.0	61.7	121	200

Table 2: GSS Transfer function frequencies as Hub Inertia varies

CONCLUSIONS

A symbolically derived analytic solution is presented for a hybrid dynamical system consisting of a rigid hub with an attached flexible appendage. The analytic solution approach is compared with the traditional computationally intense problem arising in spacecraft applications, where high-order series approximations are introduced. Closed-form solutions are obtained for the Laplace transforms for the integro-partial differential equation of motion. A generalized state space is introduced that combines the state, partial derivatives, and integral variables. The introduction of a generalized state space is seen to enable the evaluation of the s-domain spatial response in closed-form where each variable is analytically described by scalar variables. The spatial linear matrix exponential solution for the *GSS* is shown to have a very simple structure that is expected to provide critical insights for generalizing the current problem formulation to handle hub/beam translation, symmetric and antisymmetric deformational shapes, distributed control, active structures, material damping, wave motion behaviors and control approaches, as well as extensions that include beam torsion and higher-order models.

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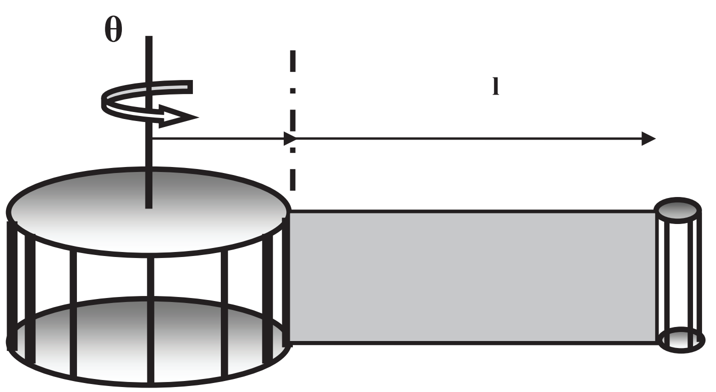

Figure 1: Hub, Beam, and Tip Mass Distributed Parameter System