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ANALYTIC CONTINUATION POWER SERIES SOLUTION FOR THE TWO-BODY PROBLEM WITH ATMOSPHERIC DRAG

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In this paper the two-body problem with atmospheric drag is considered. A cannonball drag model is utilized and the problem is solved with the analytic continuation power series technique. Recent developments of the method have made it possible to sum the series to arbitrary order enabling machine precision power series solutions for the two-body problem and the zonal gravitational perturbations. Based on these recent developments a simple drag model is considered and the corresponding recursion formulas are derived and presented. Additionally, the method will be evaluated in terms of computational cost and accuracy including a sensitivity analysis to the method parameters (number of terms in the series and step size control).

INTRODUCTION

The dynamics of the relative motion for the perturbed two-body problem is given by,

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} + \mathbf{a}_d \tag{1}$$

where, $\mathbf{r} = [x, y, z]^{\mathrm{T}}$ is the inertial relative position, $\mu = G(m_1 + m_2)$ the gravitational mass parameter, $r = \sqrt{\mathbf{r}\cdot\mathbf{r}}$ and $\mathbf{a}_d$ refers to the perturbation acceleration. For the unperturbed/classical two-body problem, $\mathbf{a}_d = \mathbf{0}$, an analytic solution exists and can be found in several works.[1,2]

For the general two-body problem, where $\mathbf{a}_d \neq \mathbf{0}$, numerical techniques are mainly used. Adaptive Runge-Kutta methods are the most widely used.[3] Several variations have been developed to handle general second-order ordinary differential and orbit propagation problems.[4–7] Gauss-Jackson method was studied extensively and compared against other numerical techniques as well as Taylor series expansion solution.[8–10] Implicit Runge-Kutta methods have also been used in astrodynamics. They are generally based on Legendre or Chebyshev polynomials combined with time-domain collocation techniques.[11–13] Furthermore, direct collocation techniques with Radial Basis Functions (RBFs) have been used to investigate general IVPs, Two Point Boundary Value Problems (TPBVPs), Optimal Control Problems (OCPs) and orbit propagation problems. The proposed methods showed fast convergence and highly accurate results.[14–16]

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Another numerical technique that has undergone significant developments in the past few years is the Modified Chebyshev-Picard Iteration (*MCPI*) method.[17,18] Since the initial introduction of *MCPI*, several contributions have been made to enhance the efficiency and the applicability of the method to a variety of astrodynamics problems. High order gravity perturbation models have been developed to capture motion of satellites near Earth.[19] Some of the enhancements that have been applied to *MCPI* include *MATLAB* libraries, variable fidelity and radially adaptive gravity approximations, segmentation/order tuning, multi-orbit accuracy and numerical stability studies, Kustaanheimo-Stiefel regularization and using the Method of Particular Solutions, *MPS*, to solve Two-Point Boundary Value Problems (TPBVPs) and Optimal Control Problems (OCPs).[20–28]

In recent years several developments have been introduced to the analytic continuation power series based solution for the two-body problem. These developments included the utilization of Leibniz product rule and two scalar kinematic variables to construct recursive relationships that are utilized to express arbitrary order series approximations.[29–32] The series solution has also been further extended to handle J_2–J_6 zonal perturbations with machine precision accuracy by implementing the backwards Horner summation scheme and a variable time-step algorithm that adapts the series order and the time-step size based on a user specified accuracy.[33]

For low Earth orbits (LEO), atmospheric drag is non-negligible. Accurate models depend on drag force, body's shape, size and orientation, and distribution of the atmosphere's density.[34] Various authors have proposed simplified models that allow for the equation of motion to be solved for in a closed form. Bertachini and Broucke used forward and backward integration with an exponential drag model to decide the region where drag should be included in swing-by trajectories.[35] Humi and Carter have found a model that helps reduce the equation of motion for cases where the radial velocity is much smaller than the tangential velocity.[36]

In this paper we extend the development of the analytic continuation technique to handle atmospheric drag as prescribed by a simple cannonball drag model. The atmospheric drag model is handled in this work by building on the highly successful recursive models developed for the zonal perturbations. Numerical results are introduced and the method is compared to an existing state of the art method, *RKN1210*. Here we successfully introduce the full analytic continuation power series solution with no truncation errors or approximations which exist in other explicit numerical integrators as the Runge-Kutta methods, without losing the low computational cost of those methods.

RECURSIVE RELATIONS IN ANALYTIC CONTINUATION

The numerical integration of the trajectory motion is achieved by introducing a Taylor series, where we can expand the position vector around a certain instant t to obtain the position at a later instant $t + \Delta t$:

$$\mathbf{r}(t + \Delta t) = \sum_{i=0}^{\infty} \frac{1}{i!}\, \mathbf{r}^{(i)}\Big|_t \Delta t^i \tag{2}$$

For small enough time increments Δt, the series is truncated at an appropriate order n to achieve a target precision

$$\mathbf{r}(t + \Delta t) \approx \sum_{i=0}^{n} \frac{1}{i!}\, \mathbf{r}^{(i)}\Big|_t \Delta t^i \tag{3}$$

An adaptive time step is defined in terms of a truncation error (i.e. last term in the Taylor series), for a target tolerance tol[32]

$$\Delta t = \left(\frac{tol\, n!}{\mathbf{r}^{(n)}}\right)^{1/n} \tag{4}$$

Higher order derivatives of $\mathbf{r}(t)$ are necessary to using this method. The following represent the recursion formulas for the arbitrary order derivative of different expressions that come in handy for the equations of motions considered in this work. Then we will present the drag model we implemented and its corresponding recursions for higher order derivatives. The recursions for the unperturbed two-body problem, as well as zonal terms J_2–J_6, have been presented elsewhere.[33]

General properties

- $a = \mathbf{u} \cdot \mathbf{v}$

$$a^{(n)} = \sum_{i=0}^{n} \binom{n}{i} \left[\mathbf{u}^{(i)} \cdot \mathbf{v}^{(n-i)}\right] \tag{5}$$

- $\mathbf{w} = \mathbf{u} \times \mathbf{v}$

$$\mathbf{w}^{(n)} = \sum_{i=0}^{n} \binom{n}{i} \left[\mathbf{u}^{(i)} \times \mathbf{v}^{(n-i)}\right] \tag{6}$$

- $f = \mathbf{u} \cdot \mathbf{u}$

$$f^{(n)} = \sum_{i=0}^{n} \binom{n}{i} \left[\mathbf{u}^{(i)} \cdot \mathbf{u}^{(n-i)}\right] \tag{7}$$

- $g_p = f^{-p/2}$

  If $f \neq 0$, one can write the definition of g_p as $f^{p/2} g_p = 1$,

$$\frac{p}{2} f^{p/2-1} \dot{f} g_p + f^{p/2} \dot{g}_p = 0 \Rightarrow \dot{f} \frac{p}{2} g_p + f \dot{g}_p = 0 \tag{8}$$

  Differentiating the last equation n times with respect to time, one obtains

$$\sum_{i=0}^{n} \binom{n}{i} \left[\frac{p}{2} f^{(i+1)} {g_p}^{(n-i)} + f^{(i)} {g_p}^{(n-i+1)}\right] = 0 \tag{9}$$

  which is used to solve recursively for ${g_p}^{(n+1)}$

$$\frac{p}{2} \dot{f} {g_p}^{(n)} + f {g_p}^{(n+1)} + \sum_{i=1}^{n} \binom{n}{i} \left[\frac{p}{2} f^{(i+1)} {g_p}^{(n-i)} + f^{(i)} {g_p}^{(n-i+1)}\right] = 0 \tag{10}$$

$${g_p}^{(n+1)} = \left(-\frac{1}{f}\right) \left(\frac{p}{2} \dot{f} {g_p}^{(n)} + \sum_{i=1}^{n} \binom{n}{i} \left[\frac{p}{2} f^{(i+1)} {g_p}^{(n-i)} + f^{(i)} {g_p}^{(n-i+1)}\right]\right) \tag{11}$$

  or

$${g_p}^{(n)} = \left(-\frac{1}{f}\right) \left(\frac{p}{2} \dot{f} {g_p}^{(n-1)} + \sum_{i=1}^{n-1} \binom{n-1}{i} \left[\frac{p}{2} f^{(i+1)} {g_p}^{(n-i-1)} + f^{(i)} {g_p}^{(n-i)}\right]\right) \tag{12}$$

- $h_p = f^{p/2}$

  The previous derivation is true for any integer p, regardless of its sign. For historic reasons g was defined as a negative power of f, and here we give a different name for the case of positive powers just to show a clear intent. Therefore,

$$h_p{}^{(n+1)} = \left(-\frac{1}{f}\right)\left(\frac{p}{2}\dot{f}h_p{}^{(n)} + \sum_{i=1}^{n}\binom{n}{i}\left[\frac{p}{2}f^{(i+1)}h_p{}^{(n-i)} - f^{(i)}h_p{}^{(n-i+1)}\right]\right) \tag{13}$$

  or

$$h_p{}^{(n)} = \left(\frac{1}{f}\right)\left(\frac{p}{2}\dot{f}h_p{}^{(n-1)} + \sum_{i=1}^{n-1}\binom{n-1}{i}\left[\frac{p}{2}f^{(i+1)}h_p{}^{(n-i-1)} - f^{(i)}h_p{}^{(n-i)}\right]\right) \tag{14}$$

- $a = Ae^{Br}$

$$\dot{a} = ABe^{Br}\dot{r} = Ba\dot{r} \tag{15}$$

$$a^{(n+1)} = B\sum_{i=0}^{n}\binom{n}{i}\left[a^{(i)}r^{(n-i+1)}\right] \tag{16}$$

  or

$$a^{(n)} = B\sum_{i=0}^{n-1}\binom{n-1}{i}\left[a^{(i)}r^{(n-i)}\right] \tag{17}$$

Cannonball drag model

The drag model we use is commonly referred to as the cannonball drag model and is given by[37]

$$\mathbf{a_{dr}} = -\rho(r)\left\|\mathbf{v}_{\text{rel}}\right\|\mathbf{v}_{\text{rel}} \tag{18}$$

where

$$\mathbf{v}_{\text{rel}} = \dot{\mathbf{r}} - \boldsymbol{\omega}_{\oplus} \times \mathbf{r} = \begin{bmatrix} \dot{x} + y\omega_{\oplus} \\ \dot{y} - x\omega_{\oplus} \\ \dot{z} \end{bmatrix} \tag{19}$$

$\omega_{\oplus} = 7.292\,115\,0 \times 10^{-5}$ rad/s is Earth's angular velocity,

$$\rho(r) = \beta\rho_{\circ}e^{-c(r-r_{\circ})/(r_f-r_{\circ})} \tag{20}$$

c is assumed to be constant per interval. The value of which can be determined by estimating the value of r_f, at which ρ_f can be evaluated,

$$c = -\log\left(\frac{\rho_f}{\rho_{\circ}}\right) \tag{21}$$

and $\beta = (1/2)C_dA/m$ is the ballistic coefficient.
The n^{th} derivative of the relative velocity vector is expressed as

$$\mathbf{v}_{\text{rel}}{}^{(n)} = \begin{bmatrix} x^{(n+1)} + y^{(n)}\omega_{\oplus} \\ y^{(n+1)} - x^{(n)}\omega_{\oplus} \\ z^{(n+1)} \end{bmatrix} \tag{22}$$

Noting that $r = h_1(\mathbf{r})$ and $\|\mathbf{v}_{\text{rel}}\| = h_1(\mathbf{v}_{\text{rel}})$, and defining $f_{\mathbf{v}} = \mathbf{v}_{\text{rel}} \cdot \mathbf{v}_{\text{rel}}$, we can use Eq. (14) with $p = 1$ thus obtaining

$$r^{(n)} = \frac{1}{f}\left(\frac{1}{2}\dot{f}r^{(n-1)} + \sum_{i=1}^{n-1}\binom{n-1}{i}\left[\frac{1}{2}f^{(i+1)}r^{(n-i-1)} - f^{(i)}r^{(n-i)}\right]\right) \tag{23}$$

$$\|\mathbf{v}_{\text{rel}}\|^{(n)} = \frac{1}{f_{\mathbf{v}}}\left(\frac{1}{2}\dot{f}_{\mathbf{v}}\|\mathbf{v}_{\text{rel}}\|^{(n-1)} + \sum_{i=1}^{n-1}\binom{n-1}{i}\left[\frac{1}{2}{f_{\mathbf{v}}}^{(i+1)}\|\mathbf{v}_{\text{rel}}\|^{(n-i-1)} - {f_{\mathbf{v}}}^{(i)}\|\mathbf{v}_{\text{rel}}\|^{(n-i)}\right]\right) \tag{24}$$

Equation (17) can be used for $\rho(r)$, for which we will define the constants $B = \beta\rho_0 e^{cr_\circ/(r_f - r_\circ)}$ and $C = -c/(r_f - r_\circ)$, hence

$$\rho(r)^{(n)} = C\sum_{i=0}^{n-1}\binom{n-1}{i}\left[\rho(r)^{(i)}r^{(n-i)}\right] \tag{25}$$

Using Leibniz rule on Eq. (18), we can write

$${\mathbf{a}_{\text{dr}}}^{(n)} = -\sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{n!}{i!j!(n-i-j)!}\rho^{(i)}\|\mathbf{v}_{\text{rel}}\|^{(j)}{\mathbf{v}_{\text{rel}}}^{(n-i-j)} \tag{26}$$

Finally, Equation (1) can be rewritten

$$\ddot{\mathbf{r}} = -\mu g_3\mathbf{r} - \rho(r)\,\|\mathbf{v}_{\text{rel}}\|\,\mathbf{v}_{\text{rel}} \tag{27}$$

and for a selected expansion order $n > 2$ we can write

$$\mathbf{r}^{(n+2)} = -\mu\sum_{i=0}^{n}\binom{n}{i}\left[{g_3}^{(i)}\mathbf{r}^{(n-i)}\right] - \sum_{i=0}^{n}\sum_{j=0}^{n-i}\frac{n!}{i!j!(n-i-j)!}\rho^{(i)}\|\mathbf{v}_{\text{rel}}\|^{(j)}{\mathbf{v}_{\text{rel}}}^{(n-i-j)} \tag{28}$$

Note that Eq. (27) is a polynomial function of variables that in turn have already been written also in polynomial forms and whose recursive formulas for derivatives of higher order are presented above. The right-hand side of Eq. (28) involves derivatives of $\mathbf{r}$ up to order $n+1$, coming from the drag term, and hence can be used to recursively obtain any higher order derivatives for the position vector, $\mathbf{r}$.

SIMULATION

In order to test the performance of the analytic continuation method for the dynamics given by Eq. (27), a LEO has been selected. The initial position and velocity vectors used are given by $\mathbf{r}_\circ = [2865400, 5191100, 2848400]^\mathsf{T}$m and $\dot{\mathbf{r}}_\circ = [-5386.2, -386.7, 6123.2]^\mathsf{T}$m/s, which correspond to a semi-major axis $a = 7.308\,961\,622 \times 10^6$ m and an eccentricity $e = 0.100$. The values used for the other model parameters are $\mu = 3.986\,004\,418 \times 10^{14}$ m^3/s^2, $\rho_\circ = 1.225$ kg/m^3, $r_\circ = 6.378\,136\,6 \times 10^6$ m, $(r_f - r_\circ)/c = 7249$ m. These values correspond to the 0–25 km range in Table 8-4 of Vallado (2008).[37]

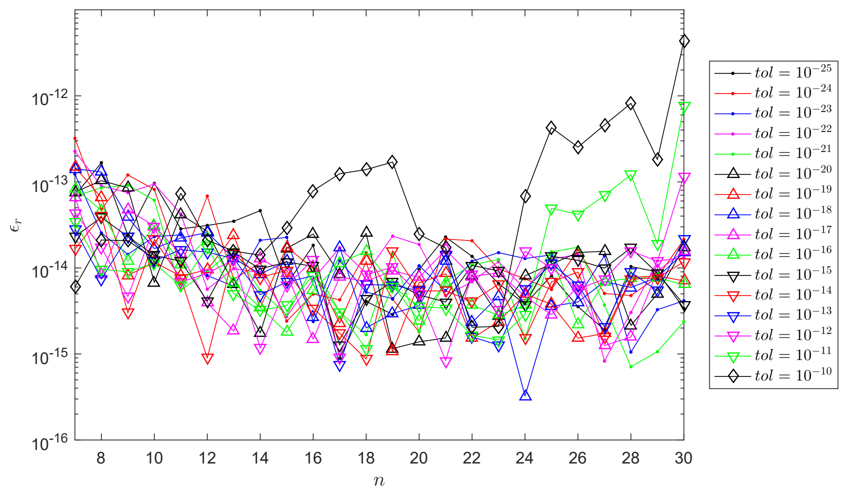

Figure 1. Round trip closure accuracy, $\epsilon_{\mathbf{r}}$, vs. expansion order, n, for different values of tol.

Two measures of performance has been taken for each simulation. The first measure is the relative error of the round trip closure (RTC),

$$\epsilon_{\mathbf{r}} = \frac{\|\tilde{\mathbf{r}}_\circ - \mathbf{r}_\circ\|}{\|\mathbf{r}_\circ\|} \tag{29}$$

where $\tilde{\mathbf{r}}_\circ$ is the final position after integrating forward in time from $t_\circ = 0\,\mathrm{s}$ to $t_f = 6218.627\,\mathrm{s}$, which corresponds to the orbit period for the unperturbed, drag-free, two-body problem with same initial conditions, and then integrating backward in time back to $t_\circ$. It is important to note here that the adaptive step size depends on the value of the highest order derivative of the position vector at the current instant, which therefore implies that the instants at which the state is evaluated when integrating forward won't exactly match with those when integrating backward. The algorithm only forces to evaluate exactly at $t_\circ$ and t_f by reducing the step size accordingly in the last iteration. The second measure of performance considered is computation time of the forward integration part, t_{cpu}.

The analytic continuation method has two tuning parameters: the expansion order n of the Taylor series to estimate at the next instant in time (Eq. (3)), and the time step size control parameter tol (Eq. (4)). Figure 1 shows $\epsilon_{\mathbf{r}}$ as a function of n for different values of tol, while Figure 2 shows $\epsilon_{\mathbf{r}}$ as a function of tol for different values of n. These figures allow us to understand the behavior of the RTC accuracy as tuning parameters change. In particular, Figure 1 makes it easy to note there is a window for $n \in [12, 30]$ and $tol < 10^{-12}$ in which most combinations of parameters yield 14 or more digits of accuracy.

Figure 3 shows a combined representation of the behavior of $\epsilon_{\mathbf{r}}$ as a function of both parameters. The absolute minimum value $\epsilon_{\mathbf{r}}^* = 3.2 \times 10^{-16}$ has been marked for reference which was obtained for $n = 24$ and $tol = 10^{-18}$, using 50 time steps and $t_{\text{cpu}} = 5\,\mathrm{ms}$ just in the forward integration section.

In Figure 4 we present the logarithm base 10 of the CPU time in milliseconds as a function of n and tol. For reference, the absolute minima CPU time observed $t_{\text{cpu}}^* = 1\,\mathrm{ms}$ are presented.

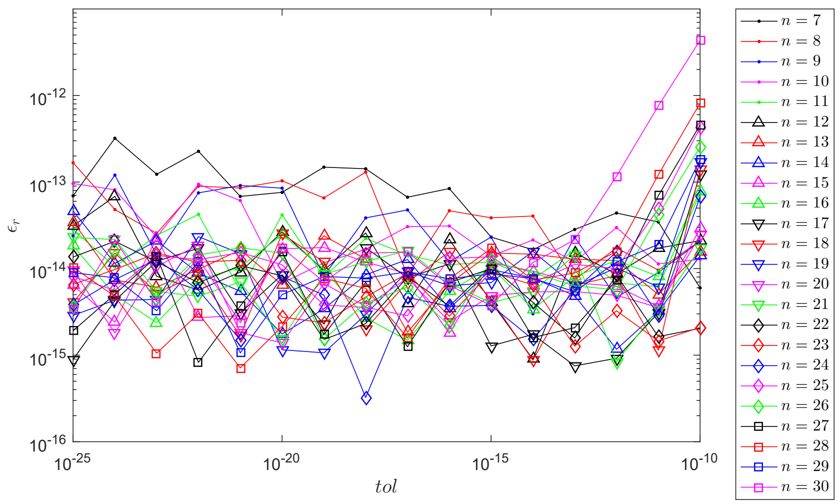

Figure 2. Round trip closure accuracy, $\epsilon_\mathbf{r}$, vs. tol, for different values of expansion order, n.

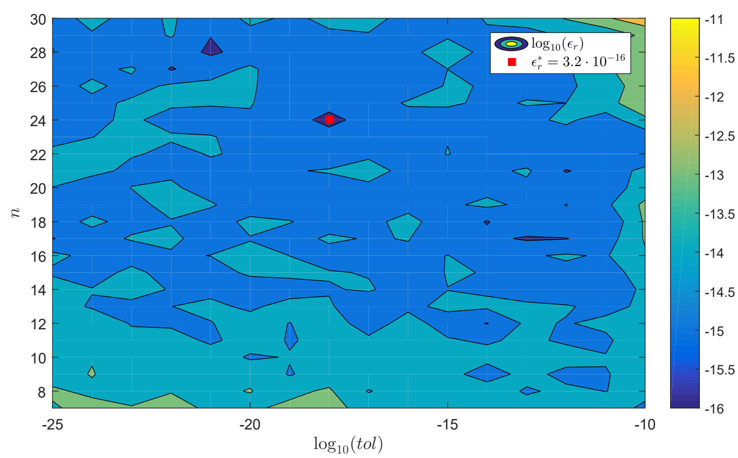

Figure 3. Round trip closure accuracy, $\epsilon_\mathbf{r}$, vs. n, and tol.

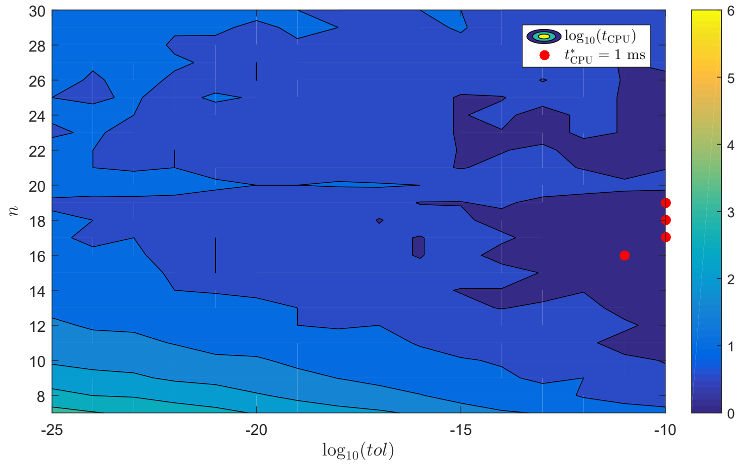

Figure 4. Round trip closure computing time measure, $\log_{10}(t_{\rm cpu})$, vs. n, and tol.

Figure 5 presents a superposition of the color maps above. This is helpful to determine the selection of parameters to better suite a particular need. For instance, $n = 16$ and $tol = 10^{-11}$ provide the best accuracy among the fastest solutions with $t_{\rm cpu} = 1\,{\rm ms}$ and $\epsilon_{\bf r} = 3.2 \times 10^{-15}$ taking 34 steps in time for the forward integration section.

Additionally, *RKN1210* with adaptive time step was used to solve the same RTC problem. With this integrator, the best performance in RTC position accuracy obtained required 166 time steps to obtain in the forward integration section with $\epsilon_{\bf r} = 3.0 \times 10^{-15}$. In order to compare RTC accuracy for similar number of time steps, we varied the tolerance parameter of *RKN1210* and obtained a solution that required 36 steps with $\epsilon_{\bf r} = 4.7 \times 10^{-13}$. This is not an exhaustive comparison, but it shows that the higher order nature of analytic continuation with respect to *RKN1210* can achieve better accuracy with similar average time steps.

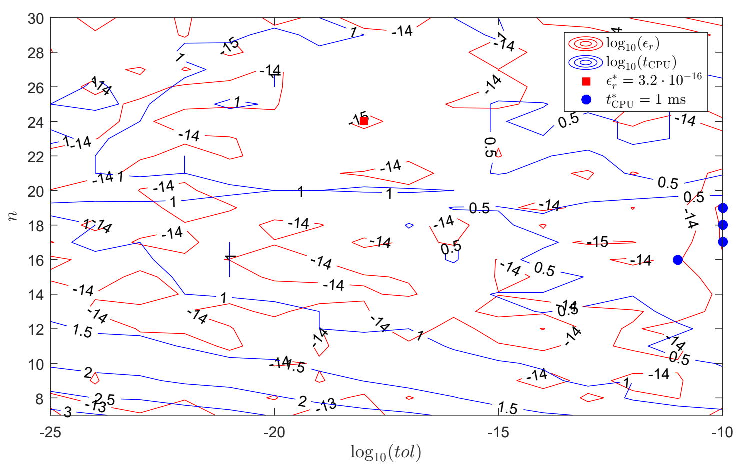

Figure 5. Round trip closure accuracy, $\epsilon_{\bf r}$, and computing time measure, $\log_{10}(t_{\rm cpu})$, vs. n, and tol.

DISCUSSION

The analytic continuation method is used to incorporate atmosphere drag in the two-body problem. The recursive method is able to produce a round trip closure with a position accuracy of 15 or more digits in a wide range of pairs of the tuning parameters. Currently the limit of accuracy of the method, for this problem, seems to be the precision used to store numbers and perform computations, opening the possibility of higher accuracies with increased precision if needed. Analytic continuation showed better results in round trip closure than *RK1210* for some similar average step size, suggesting the adaptive time step scheme and its tuning parameters for analytic continuation can prove the method to be competitive for similar problems.

The ability of the method to handle a model that includes relative velocities and exponentials of relative distances increases the library of functions that can be used in analytic continuation. High accuracies have been found here for the drag model and previously for the J_2–J_6 zonal terms. Spherical harmonics and implementations using the method of particular solutions will be studied in the near future.

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