Generalized Frequency Domain Solution for a Hybrid Rigid Hub Timoshenko Beam Rotating Aerospace Structure

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A hybrid system consisting of a rotating rigid hub and a flexible appendage following the Timoshenko beam assumptions where shear deformations are taken into account is introduced. Generalization of Lagrange's equations utilizing Hamilton's extended principle is used to derive the equations of motion and the boundary conditions of the system. Applying the Laplace transform to the integro-partial equations of motion leads to a generalized state space model for the frequency domain representation of the system. The beam sub-problem is then solved and utilized for insights for the solution of the full system. Boundary conditions at the beam free end are imposed to obtain the full solution for the state space model. The solution is used to generate transfer functions for both the rigid and the flexible modes of the system in terms of the input torque at the rigid rotating hub. No modal truncation errors are introduced into the transfer function calculations. Numerical results are presented for transfer functions frequency response using the generalized state space solution methodology.

I. Introduction

A rigid rotating body with an attached flexible beam-like appendage is a commonly used model in several engineering applications. A rotating spacecraft with a flexible solar panel is a fairly common system that utilizes such models.[1] More recently, such models are utilized in the dynamics and controls of flapping flight.[2,3] The flexible structure, usually modeled as a beam, can follow any set of deformation assumptions. figure 1 shows a hybrid system with a free end beam following the Timoshenko beam deformation assumptions which includes shear deformation as shown in figure 2. Derivation of the dynamics of such systems relies on the use of Hamilton's extended principle along with the generalization of Lagrange's equations leading to a system of hybrid, integral partial differential, equations (IPDE).[1,4–10] Solutions techniques presented in these works are mainly numerical applying finite elements methods and/or the assumed modes techniques to obtain the natural modes of the coupled system. Numerical solutions in general are approximate and the accuracy is a function of the number of elements/modes chosen which can impose a high computational cost as the need arises for more accurate results. Elgohary and Turner have recently presented a generalized state space model leading to analytic transfer functions. The resulting analytic transfer functions are derived as scalar variables by manipulating elements of the generalized state space model. The methodology has been applied to hybrid systems with the beam following Euller-Bernoulli's assumptions.[11–13] The solution is compared to the numerical assumed modes methods and found to be more accurate with no truncation errors associated with the numerical technique. Control analysis in the frequency domain utilizing the analytical transfer functions for the Euller-Bernoulli beam model has been developed and demonstrated.[14] Implementing a rest to rest maneuver for the rotating rigid hub the flexible modes of the beam are driven to stability by gains selection based on frequency domain gains and phase margins. This technique proved very useful for the Euler-Bernoulli beam model and has potential extension for other beam theories as will be shown in this work.

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In this work, the generalized state space methodology is utilized to approach the more complex Timoshenko beam model to derive analytical transfer functions that completely describe the system frequency response without the truncation that usually affects the more commonly used numerical methods. The main contribution of this work is to derive and solve a generalized state space model for a hybrid system described by a rigid hub attached to a Timoshenko beam that retains all the frequency content of the system and provides exact transfer function solutions for the system output given the input torque at the rotating hub.

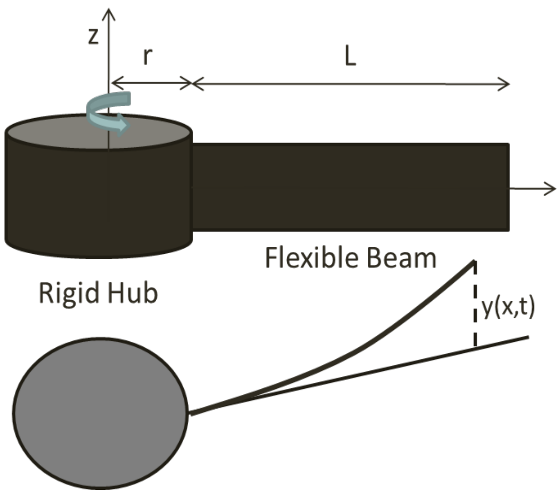

Figure 1. Hub-Beam Model

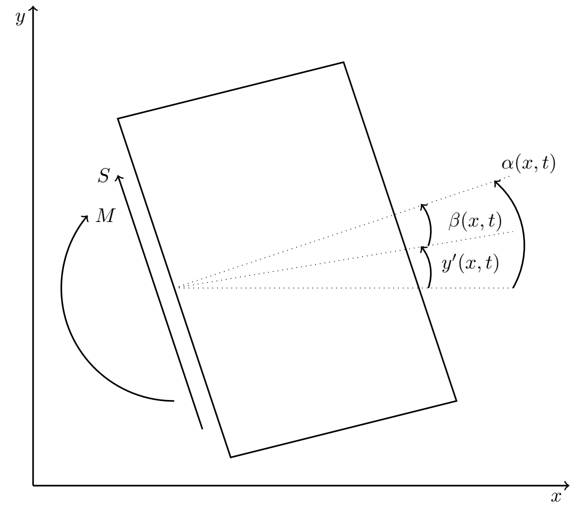

Figure 2. Deformation in Timoshenko Beam

II. Lagrange's Equations Deriving Hybrid Systems Dynamics

The derivation of the equations of motion for a hybrid system consisting of a single independent spatial variable and one deformable member (beam like structure) is presented in several works of the literature.[1,9,10] The potential and kinetic energies are given by,

$$T = T_D(\mathbf{q}, \dot{\mathbf{q}}) + \int_{l_0}^{l} \hat{T}(\text{arg})\, \mathrm{d}x + T_B(\mathbf{w}(l), \dot{\mathbf{w}}(l), \mathbf{w}'(l), \mathbf{q}, \dot{\mathbf{q}}) \tag{1}$$

$$V = V_D(\mathbf{q}, \dot{\mathbf{q}}) + \int_{l_0}^{l} \hat{V}(\text{arg})\, \mathrm{d}x + V_B(\mathbf{w}(l), \dot{\mathbf{w}}(l), \mathbf{w}'(l), \mathbf{q}, \dot{\mathbf{q}}) \tag{2}$$

where, $(*)_D$ is the discrete portion of the energy describing the rigid body, $(\hat{*})$ is the density portion of the energy, $(*)_B$ is a term describing the boundary of the energy functions, $\mathbf{q}$ and $\mathbf{w}$ are the generalized coordinates describing the rigid body motion and the flexible motion respectively and $\text{arg} = \{\mathbf{w}, \dot{\mathbf{w}}, \mathbf{w}', \mathbf{w}'', \mathbf{q}, \dot{\mathbf{q}}\}$.

The Lagrangian can then be written as,

$$\mathcal{L} = T - V = L_D(\mathbf{q}, \dot{\mathbf{q}}) + \int_{l_0}^{l} \hat{L}(\text{arg})\, \mathrm{d}x + L_B(\mathbf{w}(l), \dot{\mathbf{w}}(l), \mathbf{w}'(l), \mathbf{q}, \dot{\mathbf{q}}) \tag{3}$$

where, $L_D = T_D - V_D$, $\hat{L} = \hat{T} - \hat{V}$ and $L_B = T_B - V_B$.

The non-conservative virtual work can then be expressed as,

$$\delta W_{nc} = \mathbf{Q}^{\mathrm{T}} \delta\mathbf{q} + \int_{l_0}^{l} \hat{\mathbf{f}}^{\mathrm{T}}(x) \delta\mathbf{w}(x)\, \mathrm{d}x + \mathbf{f_1}^{\mathrm{T}} \delta\mathbf{w}(l) + \mathbf{f_2}^{\mathrm{T}} \delta\mathbf{w}'(l) \tag{4}$$

where, $\mathbf{Q}$ is the non-conservative force associated with the rigid coordinates, $\mathbf{q}$, $\hat{\mathbf{f}}$ is the non-conservative force density associated with the flex coordinates, $\mathbf{w}$, $\mathbf{f_1}$ is the non-conservative force applied at the boundary and $\mathbf{f_2}$ is the non-conservative torque applied at the boundary.

The extended Hamilton's principle is given by,

$$\int_{t_1}^{t_2} \left(\delta\mathcal{L} + \delta W_{nc}\right) \mathrm{d}x = 0 \tag{5}$$

Applying the extended Hamilton's principle and performing the analytical integration by parts.[9,10] A hybrid system of equations of motion is extracted as,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{q}}}\right) - \frac{\partial\mathcal{L}}{\partial\mathbf{q}} = \mathbf{Q}^{\mathrm{T}} \\
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\hat{L}}{\partial\dot{\mathbf{w}}}\right) - \frac{\partial\hat{L}}{\partial\mathbf{w}} + \frac{\partial}{\partial x}\left(\frac{\partial\hat{L}}{\partial\mathbf{w}'}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial\hat{L}}{\partial\mathbf{w}''}\right) = \hat{\mathbf{f}}^{\mathrm{T}}
\end{aligned} \tag{6}$$

The boundary conditions are also derived from Hamilton's principle and are given by,

$$\begin{aligned}
&\left[\frac{\partial\hat{L}}{\partial\mathbf{w}'} - \frac{\partial}{\partial x}\left(\frac{\partial\hat{L}}{\partial\mathbf{w}''}\right)\right]\delta\mathbf{w}\Bigg|_{l_0}^{l} + \left[\frac{\partial L_B}{\partial\mathbf{w}(l)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_B}{\partial\mathbf{w}'(l)}\right)\right]\delta\mathbf{w}(l) + \mathbf{f_1}^{\mathrm{T}}\delta\mathbf{w}(l) = 0 \\
&\frac{\partial\hat{L}}{\partial\mathbf{w}''}\delta\mathbf{w}'\Bigg|_{l_0}^{l} + \left[\frac{\partial L_B}{\partial\mathbf{w}'(l)} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_B}{\partial\dot{\mathbf{w}}'(l)}\right)\right]\delta\mathbf{w}'(l) + \mathbf{f_2}^{\mathrm{T}}\delta\mathbf{w}'(l) = 0
\end{aligned} \tag{7}$$

From the above development the equations of motion of a hybrid system with one spatial independent variable and a single flexible body can be derived along with its associated boundary conditions. Next, those steps are applied to derive the equations of motion for the system of interest shown in figure 1 while accounting for shear deformations of the Timoshenko model shown in figure 2.

III. System Equations and the Generalized State Space Model

The kinetic and potential energies of the system in figure 1 are given by,[1]

$$\begin{aligned}
T &= T_{\text{hub}} + T_{\text{appendage}} \\
T &= \frac{1}{2}I_{\text{hub}}\dot{\theta}^2 + \frac{1}{2}\int_0^L \rho\left(\dot{y} + (x+r)\dot{\theta}\right)^2 + \left(\frac{\rho I}{A}\right)\left(\dot{\alpha} + \dot{\theta}\right)^2 \mathrm{d}x
\end{aligned} \tag{8}$$

$$V = \frac{1}{2}\int_0^L EI\left(\alpha'\right)^2 + KGA\left(\alpha - y'\right)^2 \mathrm{d}x \tag{9}$$

where, E is the beam Young's modulus, I the moment of inertia of the cross section about the centroid axis, ρ the beam mass per unit length, k the shear coefficient, G the modulus of rigidity, A the area of the beam cross section, r the radius of the rigid hub, L the length of the flexible appendage, I_{hub} moment of inertia of the rigid hub, θ the hub rotation, y the beam deformation and α the rotation of the cross section for shear deformation. The Lagrangian is then expressed as,

$$\begin{aligned}
\mathcal{L} &= T - V \\
&= \frac{1}{2}I_{\text{hub}}\dot{\theta}^2 + \frac{1}{2}\int_0^L \rho\left(\dot{y} + (x+r)\dot{\theta}\right)^2 + \left(\frac{\rho I}{A}\right)\left(\dot{\alpha} + \dot{\theta}\right)^2 - EI\left(\alpha'\right)^2 - KGA\left(\alpha - y'\right)^2 \mathrm{d}x
\end{aligned} \tag{10}$$

Following the same notation in the generalized derivation presented earlier, the hybrid system coordinates are expressed as,

$$\mathbf{q}(t) = \theta \qquad \mathbf{w}(x,t) = [y,\, \alpha]^{\mathrm{T}} \tag{11}$$

With the forces given by,

$$\mathbf{Q} = u \qquad \hat{\mathbf{f}} = 0 \qquad \mathbf{f_1} = \mathbf{f_2} = 0 \tag{12}$$

Applying Lagrange's generalized equations and Hamilton's extended principle, the hybrid system governing equations are derived as,

$$
\begin{aligned}
& I_{\text{hub}}\ddot{\theta} + \int_0^L \rho(x+r)\left(\ddot{y} + (x+r)\ddot{\theta}\right) + \left(\frac{\rho I}{A}\right)\left(\ddot{\alpha} + \ddot{\theta}\right) \,\mathrm{d}x = u \\
& \rho\left(\ddot{y} + (x+r)\ddot{\theta}\right) + KGA\left(\alpha' - y''\right) = 0 \\
& \frac{\rho I}{A}\left(\ddot{\alpha} + \ddot{\theta}\right) + KGA\left(\alpha - y'\right) - EI\alpha'' = 0
\end{aligned}
\tag{13}
$$

With no boundary dependent terms in the Lagrangian, the boundary conditions for the system are,

$$
\begin{array}{lllll}
\text{at} & x = 0 & y = 0 & , & \alpha = 0 \\
\text{at} & x = L & EI\alpha'|_L = 0 & , & KGA\left(\alpha - y'\right)|_L = 0
\end{array}
\tag{14}
$$

Taking the Laplace transform, represented by μ, for Eq. (13) and performing integration by parts to remove the spatial dependency from the integral term, the first equation in Eq. (13) representing the rigid body motion can be expressed as,

$$
\mu^2 I_{\text{hub}}\bar{\theta} + \mu^2\rho\left[(x+r)\int \bar{y}\,\mathrm{d}x - \iint \bar{y}\,\mathrm{d}x\,\mathrm{d}x\right] + \frac{\mu^2\rho}{3}\left[(L+r)^3 - r^3\right]\bar{\theta} + \mu^2\frac{\rho I}{A}\int \bar{\alpha}\,\mathrm{d}x + \mu^2\frac{\rho I}{A}L\bar{\theta} = \bar{u} \tag{15}
$$

A Generalized State Space (GSS) model is developed as,

$$
\begin{array}{ll}
z_1 = \iint \bar{y}\,\mathrm{d}x\,\mathrm{d}x & z_1' = z_3 \\
z_2 = \int \bar{\alpha}\,\mathrm{d}x & z_2' = z_4 \\
z_3 = \int \bar{y}\,\mathrm{d}x & z_3' = z_5 \\
z_4 = \bar{\alpha} & z_4' = z_6 \\
z_5 = \bar{y} & z_5' = z_7 \\
z_6 = \bar{\alpha}' & z_6' = \left(\frac{\mu^2\rho}{EA} + \frac{KGA}{EI}\right)\bar{\alpha} + \frac{\mu^2\rho}{EA}\bar{\theta} - \frac{KGA}{EI}\bar{y}' = (\psi + \Gamma)\,z_4 + \psi\bar{\theta} - \Gamma z_7 \\
z_7 = \bar{y}' & z_7' = \frac{\mu^2\rho}{KGA}\left(\bar{y} + x\bar{\theta}\right) + \bar{\alpha}' = \beta\left(z_5 + x\bar{\theta}\right) + z_6
\end{array}
\tag{16}
$$

where, $\beta \equiv \frac{\mu^2}{KGA}$, $\psi \equiv \frac{\mu^2\rho}{EA}$ and the constant $\Gamma \equiv \frac{KGA}{EI}$. The state space is generalized in the sense that the states consist of variable, spatial partial derivatives of variables, and first and second order integrals of variables, which mix solutions at points in the flexible body domain with global response variables. In terms of the GSS variables the boundary conditions are defined as,

$$
\begin{aligned}
& \mathbf{z}(0) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & z_6 & z_7 \end{bmatrix}^{\mathrm{T}} \\
& z_6(L) = 0 \qquad z_4(L) - z_7(L) = 0
\end{aligned}
\tag{17}
$$

The state space form of Eq. (16) is represented as,

$$
\{\mathbf{z}'\} = \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & (\psi+\Gamma) & 0 & 0 & -\Gamma \\
0 & 0 & 0 & 0 & \beta & 1 & 0
\end{bmatrix}\{\mathbf{z}\} + \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \psi\bar{\theta} \\ \beta x\bar{\theta} \end{Bmatrix} = [A]\{\mathbf{z}\} + \mathbf{b} \tag{18}
$$

where only constant coefficient matrix and vector components appear.

IV. Solution of the Generalized State Space Model

Equation (18) has the well known solution,

$$\mathbf{z}(x) = \exp(Ax)\mathbf{z}(0) + \int_0^L \exp\left(A(L-\tau)\right)\mathbf{b}(\tau)\,\mathrm{d}\tau \tag{19}$$

The solution of Eq. (19) is realized once a solution is provided for computing the matrix exponential terms. To this end, a two-step process is introduced. First, by examining the eigenvalues of the beam sub-problem

$$\begin{aligned} \mathbf{q}' &= C\mathbf{q} \\ C &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \psi+\beta & 0 & 0-\Gamma & \\ 0 & \beta & 1 & 0 \end{bmatrix} \end{aligned} \tag{20}$$

The eigenvalues for the sub-matrix are obtained as,

$$\Lambda = \mathrm{Diag}\left\{ \begin{matrix} \frac{1}{2}\sqrt{2\beta+2\psi+2\sqrt{\beta^2-2\beta\psi+\psi^2-4\beta\Gamma}} \\ -\frac{1}{2}\sqrt{2\beta+2\psi+2\sqrt{\beta^2-2\beta\psi+\psi^2-4\beta\Gamma}} \\ \frac{1}{2}\sqrt{2\beta+2\psi-2\sqrt{\beta^2-2\beta\psi+\psi^2-4\beta\Gamma}} \\ -\frac{1}{2}\sqrt{2\beta+2\psi-2\sqrt{\beta^2-2\beta\psi+\psi^2-4\beta\Gamma}} \end{matrix} \right\} \tag{21}$$

C can then be expressed as $C = V\Lambda V^{-1}$ where V are the eigenvectors of C. By using this eigen decomposition technique the matrix exponential of C needed in Eq. (19) is computed as

$$\exp(Cx) = V\exp(\Lambda x)V^{-1} \tag{22}$$

By defining the recurring expressions obtained from the beam sub-problem eigenvalues as,

$$\begin{aligned} \alpha &\equiv \sqrt{\beta^2-2\beta\psi+\psi^2-4\beta\Gamma} \\ \lambda &\equiv \sqrt{2\alpha+2\beta+2\psi} \\ \eta &\equiv \sqrt{-2\alpha+2\beta+2\psi} \end{aligned} \tag{23}$$

The matrix exponential problem in Eq. (19) is solved via symbolic manipulations by expressing the full solution in terms of the recurring expressions in Eq. (23). The resultant is further simplified as a set of hyperbolic functions and their coefficients for each element of the matrix exponential,

$$[\exp(Ax)]_{1\ldots 7,1\ldots 4} =$$

$$\begin{bmatrix} 1 & 0 & x & p_1\sinh\left(\frac{1}{2}\lambda x\right)+p_2\sinh\left(\frac{1}{2}\eta x\right)+p_3 & \cosh\left(\frac{1}{2}\eta x\right)p_5+\cosh\left(\frac{1}{2}\lambda x\right)p_4+p_6 \\ 0 & 1 & 0 & p_{13}\sinh\left(\frac{1}{2}\lambda x\right)+p_{14}\sinh\left(\frac{1}{2}\eta x\right) & \cosh\left(\frac{1}{2}\eta x\right)p_{16}+\cosh\left(\frac{1}{2}\lambda x\right)p_{15}+p_{17} \\ 0 & 0 & 1 & \cosh\left(\frac{1}{2}\eta x\right)p_{24}+\cosh\left(\frac{1}{2}\lambda x\right)p_{23}+p_{25} & p_{26}\sinh\left(\frac{1}{2}\lambda x\right)+p_{27}\sinh\left(\frac{1}{2}\eta x\right) \\ 0 & 0 & 0 & \cosh\left(\frac{1}{2}\eta x\right)p_{34}+\cosh\left(\frac{1}{2}\lambda x\right)p_{33} & p_{35}\sinh\left(\frac{1}{2}\lambda x\right)+p_{36}\sinh\left(\frac{1}{2}\eta x\right) \\ 0 & 0 & 0 & p_{41}\sinh\left(\frac{1}{2}\lambda x\right)+p42\sinh\left(\frac{1}{2}\eta x\right) & \cosh\left(\frac{1}{2}\eta x\right)p_{44}+\cosh\left(\frac{1}{2}\lambda x\right)p_{43} \\ 0 & 0 & 0 & p_{49}\sinh\left(\frac{1}{2}\lambda x\right)+p_{50}\sinh\left(\frac{1}{2}\eta x\right) & \cosh\left(\frac{1}{2}\eta x\right)p_{52}+\cosh\left(\frac{1}{2}\lambda x\right)p_{51} \\ 0 & 0 & 0 & \cosh\left(\frac{1}{2}\eta x\right)p_{58}+\cosh\left(\frac{1}{2}\lambda x\right)p_{57} & p_{59}\sinh\left(\frac{1}{2}\lambda x\right)+p_{60}\sinh\left(\frac{1}{2}\eta x\right) \end{bmatrix} \tag{24}$$

$$[\exp(Ax)]_{1\ldots 7,6\ldots 7} =$$

$$\begin{bmatrix} \cosh\left(\frac{1}{2}\eta x\right)p_8+\cosh\left(\frac{1}{2}\lambda x\right)p_7+p_9 & p_{10}\sinh\left(\frac{1}{2}\lambda x\right)+p_{11}\sinh\left(\frac{1}{2}\eta x\right)+p_{12} \\ \cosh\left(\frac{1}{2}\eta x\right)p_{19}+\cosh\left(\frac{1}{2}\lambda x\right)p_{18}+p_{20} & p_{21}\sinh\left(\frac{1}{2}\lambda x\right)+p_{22}\sinh\left(\frac{1}{2}\eta x\right) \\ p_{28}\sinh\left(\frac{1}{2}\lambda x\right)+p_{29}\sinh\left(\frac{1}{2}\eta x\right) & \cosh\left(\frac{1}{2}\eta x\right)p_{31}+\cosh\left(\frac{1}{2}\lambda x\right)p_{30}+p_{32} \\ p_{37}\sinh\left(\frac{1}{2}\lambda x\right)+p_{38}\sinh\left(\frac{1}{2}\eta x\right) & \cosh\left(\frac{1}{2}\eta x\right)p_{40}+\cosh\left(\frac{1}{2}\lambda x\right)p_{39} \\ \cosh\left(\frac{1}{2}\eta x\right)p_{46}+\cosh\left(\frac{1}{2}\lambda x\right)p_{45} & p_{47}\sinh\left(\frac{1}{2}\lambda x\right)+p_{48}\sinh\left(\frac{1}{2}\eta x\right) \\ \cosh\left(\frac{1}{2}\eta x\right)p_{54}+\cosh\left(\frac{1}{2}\lambda x\right)p_{53} & p_{55}\sinh\left(\frac{1}{2}\lambda x\right)+p_{56}\sinh\left(\frac{1}{2}\eta x\right) \\ p_{61}\sinh\left(\frac{1}{2}\lambda x\right)+p_{62}\sinh\left(\frac{1}{2}\eta x\right) & \cosh\left(\frac{1}{2}\eta x\right)p_{64}+\cosh\left(\frac{1}{2}\lambda x\right)p_{63} \end{bmatrix}$$

The p coefficients result from the symbolic generation of the matrix exponential. These expressions are provided in detail in the Appendix. With the full analytical solution of the matrix exponential available, Eq. (24), the homogeneous and the forced solutions of the GSS system is obtained as,

$$Z_H \equiv \exp(Ax)\mathbf{z}(0)$$

$$Z_H = \begin{bmatrix} \left(\cosh\left(\frac{1}{2}\eta x\right)p_8+\cosh\left(\frac{1}{2}\lambda x\right)p_7+p_9\right)z_6+\left(p_{10}\sinh\left(\frac{1}{2}\lambda x\right)+p_1 1\sinh\left(\frac{1}{2}\eta x\right)+p_{12}\right)z_7 \\ \left(\cosh\left(\frac{1}{2}\eta x\right)p_{19}+\cosh\left(\frac{1}{2}\lambda x\right)p_{18}+p_{20}\right)z_6+\left(p_{21}\sinh\left(\frac{1}{2}\lambda x\right)+p_{22}\sinh\left(\frac{1}{2}\eta x\right)\right)z_7 \\ \left(p_{28}\sinh\left(\frac{1}{2}\lambda x\right)+p_{29}\sinh\left(\frac{1}{2}\eta x\right)\right)z_6+\left(\cosh\left(\frac{1}{2}\eta x\right)p_{31}+\cosh\left(\frac{1}{2}\lambda x\right)p_{30}+p_{32}\right)z_7 \\ \left(p_{37}\sinh\left(\frac{1}{2}\lambda x\right)+p_{38}\sinh\left(\frac{1}{2}\eta x\right)\right)z_6+\left(\cosh\left(\frac{1}{2}\eta x\right)p_{40}+\cosh\left(\frac{1}{2}\lambda x\right)p_{39}\right)z_7 \\ \left(\cosh\left(\frac{1}{2}\eta x\right)p_{46}+\cosh\left(\frac{1}{2}\lambda x\right)p_{45}\right)z_6+\left(p_{47}\sinh\left(\frac{1}{2}\lambda x\right)+p_{48}\sinh\left(\frac{1}{2}\eta x\right)\right)z_7 \\ \left(\cosh\left(\frac{1}{2}\eta x\right)p_{54}+\cosh\left(\frac{1}{2}\lambda x\right)p_{53}\right)z_6+\left(p_{55}\sinh\left(\frac{1}{2}\lambda x\right)+p_{56}\sinh\left(\frac{1}{2}\eta x\right)\right)z_7 \\ \left(p_{61}\sinh\left(\frac{1}{2}\lambda x\right)+p_{62}\sinh\left(\frac{1}{2}\eta x\right)\right)z_6+\left(\cosh\left(\frac{1}{2}\eta x\right)p_{64}+\cosh\left(\frac{1}{2}\lambda x\right)p_{63}\right)z_7 \end{bmatrix} \tag{25}$$

The forced solution is then derived as,

$$Z_F \equiv \int_0^L \exp\left(A(L-\tau)\right)\mathbf{b}(\tau)\,\mathrm{d}\tau$$

$$Z_F = \bar{\theta}\int_0^L \exp\left(A(L-\tau)\right)\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \psi \\ \beta\tau \end{Bmatrix}\mathrm{d}\tau = \begin{bmatrix} I_1 \\ I_2 \\ I_3 \\ I_4 \\ I_5 \\ I_6 \\ I_7 \end{bmatrix}\bar{\theta} \tag{26}$$

where the exact expressions for $I_1, \ldots, I_7$ are not shown here for the sake of brevity and can be easily obtained with any symbolic math software. Applying the boundary conditions at the beam end provides the required equations to evaluate the initial z_6, z_7 and obtain the full solution for the GSS system of equations.

$$\begin{aligned}
z_6(L) &= 0 \\
&= \left(\cosh\left(\frac{1}{2}\eta L\right) p_{54} + \cosh\left(\frac{1}{2}\lambda L\right) p_{53}\right) z_6 + \left(p_{55}\sinh\left(\frac{1}{2}\lambda L\right) + p_{56}\sinh\left(\frac{1}{2}\eta L\right)\right) z_7 + I_6\bar{\theta} \\
z_4(L) - z_7(L) &= 0 \\
&= \left(p_{37}\sinh\left(\frac{1}{2}\lambda x\right) + p_{38}\sinh\left(\frac{1}{2}\eta x\right)\right) z_6 + \left(\cosh\left(\frac{1}{2}\eta x\right) p_{40} + \cosh\left(\frac{1}{2}\lambda x\right) p_{39}\right) z_7 + I_4\bar{\theta} \\
&- \left[\left(p_{61}\sinh\left(\frac{1}{2}\lambda x\right) + p_{62}\sinh\left(\frac{1}{2}\eta x\right)\right) z_6 + \left(\cosh\left(\frac{1}{2}\eta x\right) p_{64} + \cosh\left(\frac{1}{2}\lambda x\right) p_{63}\right) z_7 + I_7\bar{\theta}\right]
\end{aligned} \tag{27}$$

Equation (27) represents a linear set of equations to solve for z_6, z_7 and complete the full analytical GSS solution as,

$$\begin{Bmatrix} z_6 \\ z_7 \end{Bmatrix} = \begin{bmatrix} \cosh\left(\frac{1}{2}\eta L\right) p_{54} + \cosh\left(\frac{1}{2}\lambda L\right) p_{53} & p_{55}\sinh\left(\frac{1}{2}\lambda L\right) + p_{56}\sinh\left(\frac{1}{2}\eta L\right) \\ (p_{37} - p_{61})\sinh\left(\frac{1}{2}\lambda x\right) + (p_{38} - p_{62})\sinh\left(\frac{1}{2}\eta x\right) & \cosh\left(\frac{1}{2}\eta x\right)(p_{40} - p_{64}) + \cosh\left(\frac{1}{2}\lambda x\right)(p_{39} - p_{63}) \end{bmatrix}^{-1} \begin{Bmatrix} -I_6 \\ I_7 - I_4 \end{Bmatrix} \bar{\theta} \tag{28}$$

This yields the full solution for the GSS model in Eq. (18), where the generalized state variables are expressed as a function of the defined system parameters, β, ψ, Γ and the dependent variable x with $\bar{\theta}$ appearing linearly in the solution as a result from evaluating the convolution integral in Eq. (19) as,

$$\mathbf{z} = \mathbf{g}(x, \beta, \psi, \Gamma)\,\bar{\theta} \tag{29}$$

where, $\mathbf{g} \equiv \begin{bmatrix} g_1 & g_2 & g_3 & g_4 & g_5 & g_6 & g_7 \end{bmatrix}^{\mathrm{T}}$.

V. System Transfer Functions & Frequency Response Results

As shown in Eq. (29) the state variables of the GSS model are expressed in terms of the function $\mathbf{g}$. From Eq. (15) the transfer function for $\bar{\theta}$ in terms of the input torque $\bar{u}$ is expressed as,

$$G_1 \equiv \frac{\bar{\theta}}{\bar{u}} = \frac{1}{\mu^2\left[J + \rho(x+r)g_3 - \rho g_1 + \frac{\rho I}{A} g_2\right]} \tag{30}$$

where, $J \equiv I_{\text{hub}} + \frac{\rho}{3}\left((L+r)^3 - r^3\right)$ is the generalized inertia expression. The transfer functions for $\bar{y}$ and $\bar{\alpha}$ can then be obtained from

$$\begin{aligned}
\bar{y} &= z_5 = g_5\bar{\theta} = g_5 G_1 \bar{u} \\
\bar{\alpha} &= z_4 = g_4\bar{\theta} = g_4 G_1 \bar{u}
\end{aligned} \tag{31}$$

Hence the appendage deflection and shear angle transfer functions can be obtained from the input torque as,

$$\begin{aligned}
G_2 &\equiv \frac{\bar{y}}{\bar{u}} = \frac{g_5}{\mu^2\left[J + \rho(x+r)g_3 - \rho g_1 + \frac{\rho I}{A} g_2\right]} \\
G_3 &\equiv \frac{\bar{\alpha}}{\bar{u}} = \frac{g_4}{\mu^2\left[J + \rho(x+r)g_3 - \rho g_1 + \frac{\rho I}{A} g_2\right]}
\end{aligned} \tag{32}$$

This development produces a full analytical expression for the system flexible modes in terms of the input torque at the rigid hub. The flex response natural modes can then be extracted from the above analytical transfer functions with no truncation that usually affects numerical approximations. As a numerical example the frequency response of the system transfer functions are evaluated for the set of physical parameters in table 1

Table 1. System Parameters Values

Parameter	Value
I_{hub}	8 slug-ft^2
ρ	0.0271875 slug/ft
E	0.1584×10^{10} lb/ft^2
L	4 ft
r	1 ft
I	0.47095×10^{-7} ft^4
m	0.1569 slug
I_{tip}	0.0018 slug-ft^2
K	5/6
ν	0.3
A	7.5176×10^{-4} ft ^{2}

where the modulus of rigidity G can be obtained from Young's modulus and Poisson's ratio as $G = \frac{E}{2(1+nu)}$. Substituting with $\mu = j\omega$ into the transfer functions in Eq. (30) and Eq. (32) the frequency response of the hybrid system is obtained. Figure 3 throught figure 5 show the frequency response for $G_1(j\omega)$, $G_2(j\omega)$ and $G_3(j\omega)$, respectively.

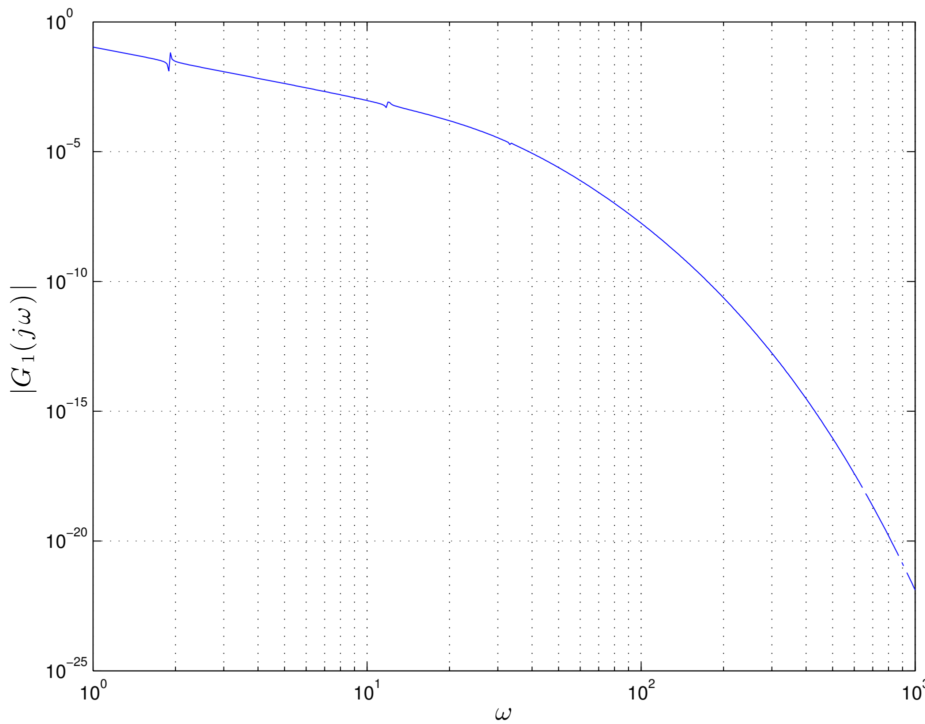

Figure 3. G_1 **Frequency Response**

As shown the complete frequency response of the system transfer functions is obtained. The flexible modes experience resonance at their natural frequencies which can be notched out when considering controlling the flexible appendage. This development can also be extended to address the control problem of the hybrid system in the frequency domain to drive the rigid state while controlling the resonant modes of the flexible

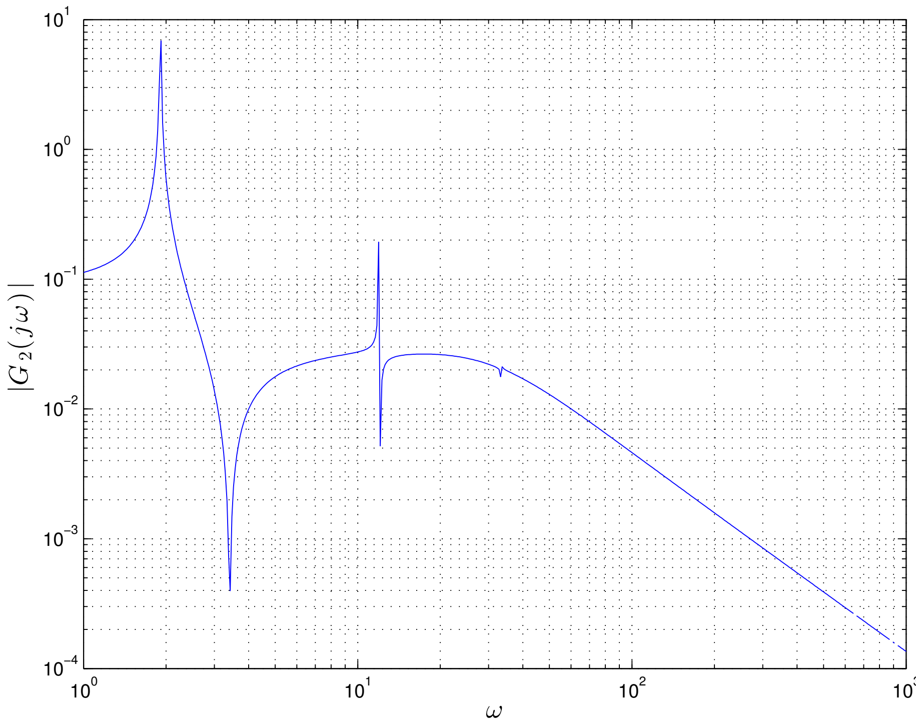

Figure 4. G_2 Frequency Response

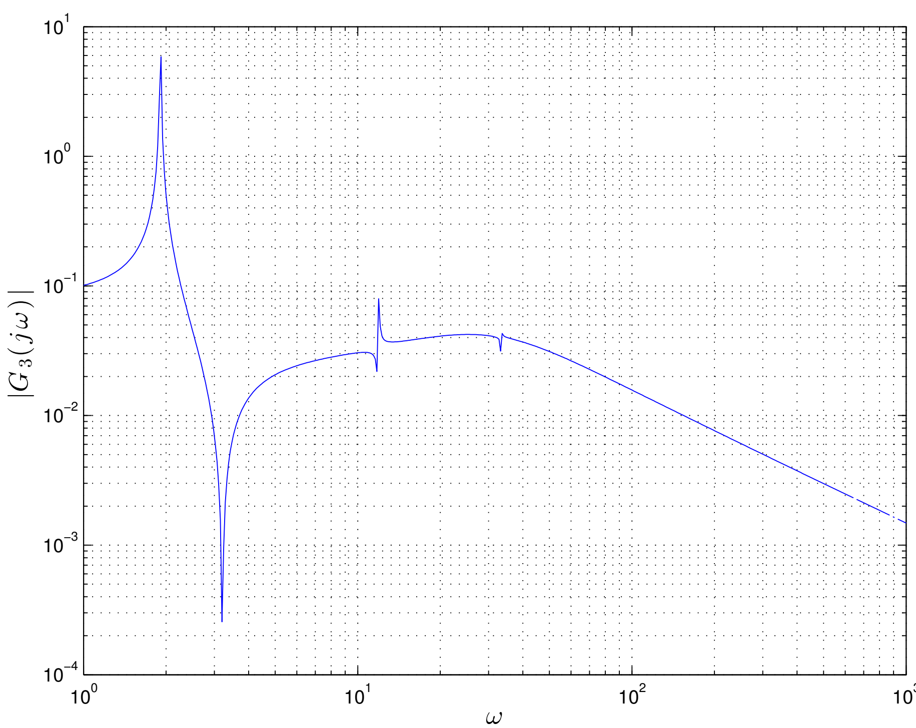

Figure 5. G_3 Frequency Response

states.

VI. Discussion & Conclusion

The generalized state space approach provided closed form solution for the system frequency response for Timoshenko beam model. The coupled bending/shear beam sub-problem is first used to obtain the eigenvalues and identify the form of the block structure of the matrix exponential solution. The homogeneous and forced solution of the GSS model are obtained from the matrix exponential analytical expressions. The GSS is then fully solved by applying the boundary conditions at the beam end and evaluating the unknown initial conditions. By extracting the elements of the GSS model and using the original system of equations of motion, analytical transfer functions are obtained form the system output in terms of the input torque at the rotating rigid hub. It is important to note that only scalar operations are required to generate the system transfer function calculations. Several other boundary conditions can be applied, e.g. beam with tip mass, and the same steps presented here can be followed in order to obtain the full analytical solutions for the transfer functions. A set of physical parameters are introduced and the frequency response is obtained. By utilizing the full transfer function solution provided by the GSS approach any control problem design in the frequency domain can be addressed. A general control objective would be to drive the rigid hub through a rest to rest maneuver while controlling the natural modes of the flexible appendage(s).

The presented solution methodology is accurate and provides a means for control systems design in the frequency domain. As an extension to this study the analytical solution is to be compared with existing numerical methods like the assumed modes methods using eigenfunctions of the clamped-free Timoshenko beam. The control problem in the frequency domain by using the frequency response and the phase angle information for gain selection to meet a specified control objective will also be studied and presented in future works.

Appendix

A. Coefficients of the Matrix Exponential Solution

In this section the coefficients of the matrix exponential solution are fully expressed in terms of β, ψ, Γ and the predefined variables in Eq. (23). First the following two quantities are defined.

$$\begin{aligned} D_1 &\equiv \frac{\alpha\lambda^4\eta^8}{64} \\ D_2 &\equiv \frac{-\alpha\lambda^5\eta^{11}}{128} \end{aligned} \tag{33}$$

The set of coefficients in the first row of the matrix exponential solution, $p_1, \ldots, p_{12}$ can then be expressed as,

$$
\begin{aligned}
p_1 &= \frac{-\alpha\,\eta + \beta\,\eta + \eta\,\psi}{\beta\alpha\lambda\eta} \\
p_2 &= \frac{-\alpha\,\lambda - \beta\,\lambda - \lambda\,\psi}{\beta\alpha\lambda\eta} \\
p_3 &= \frac{x}{\beta} \\
p_4 &= \frac{-32\,\beta}{D_1}\left[2\,\Gamma^3\beta^2 + \Gamma^2\beta^2\alpha + 2\,\Gamma^2\beta\,\psi\,\alpha - \Gamma^2\beta^3 + \Gamma^2\beta^2\psi - 4\,\Gamma^2\beta\,\psi^2 + \Gamma\beta^2\psi\,\alpha + 2\,\Gamma\beta\,\psi^2\alpha \right. \\
&\left. -\Gamma\psi^3\alpha - \Gamma\beta^3\psi - \Gamma\beta^2\psi^2 - 5\,\Gamma\beta\,\psi^3 + \Gamma\psi^4 - \psi^4\alpha - \beta\,\psi^4 + \psi^5\right] \\
p_5 &= \frac{32\,\beta^3}{D_1}\left[2\,\Gamma^3 + \Gamma^2\alpha - \Gamma^2\beta + 5\,\Gamma^2\psi + 2\,\Gamma\alpha\,\psi - 2\,\Gamma\beta\,\psi + 4\,\Gamma\psi^2 + \alpha\,\psi^2 - \beta\,\psi^2 + \psi^3\right] \\
p_6 &= \frac{32\,\beta\,\psi}{D_1}\left[2\,\Gamma^2\beta\,\alpha - 4\,\Gamma^2\beta^2 - 4\,\Gamma^2\beta\,\psi - \alpha\,\Gamma\beta^2 + 2\,\alpha\,\Gamma\beta\,\psi - \Gamma\psi^2\alpha + \Gamma\beta^3 - 5\,\Gamma\beta^2\psi - 5\,\beta\,\Gamma\psi^2 \right. \\
&\left. +\Gamma\psi^3 - \alpha\,\beta^2\psi - \alpha\,\psi^3 + \beta^3\psi - \beta^2\psi^2 - \beta\,\psi^3 + \psi^4\right] \\
p_7 &= \frac{32\,\beta}{D_1}\left[\Gamma^2\beta\,\alpha - 3\,\Gamma^2\beta^2 - 3\,\Gamma^2\beta\,\psi - \alpha\,\Gamma\beta^2 - \Gamma\psi^2\alpha + \Gamma\beta^3 \right. \\
&\left. -3\,\Gamma\beta^2\psi - 3\,\beta\,\Gamma\psi^2 + \Gamma\psi^3 - \alpha\,\beta^2\psi - \alpha\,\beta\,\psi^2 - \alpha\,\psi^3 + \beta^3\psi + \psi^4\right] \\
p_8 &= \frac{32\,\beta^2}{D_1}\left[\Gamma^2\alpha - \Gamma^2\beta - \Gamma^2\psi + 2\,\Gamma\alpha\,\psi - 2\,\Gamma\beta\,\psi - 2\,\Gamma\psi^2 + \alpha\,\psi^2 - \beta\,\psi^2 - \psi^3\right] \\
p_9 &= \frac{-32\,\beta}{D_1}\left[2\,\Gamma^2\beta\,\alpha - 4\,\Gamma^2\beta^2 - 4\,\Gamma^2\beta\,\psi - \alpha\,\Gamma\beta^2 + 2\,\alpha\,\Gamma\beta\,\psi - \Gamma\psi^2\alpha + \Gamma\beta^3 - 5\,\Gamma\beta^2\psi - 5\,\beta\,\Gamma\psi^2 \right. \\
&\left. +\Gamma\psi^3 - \alpha\,\beta^2\psi - \alpha\,\psi^3 + \beta^3\psi - \beta^2\psi^2 - \beta\,\psi^3 + \psi^4\right] \\
p_{10} &= \frac{128\,\beta\,\eta}{D_2}\left[2\,\Gamma^4\beta^2 + \Gamma^3\alpha\,\beta^2 + 2\,\Gamma^3\alpha\,\beta\,\psi - \Gamma^3\beta^3 + 3\,\Gamma^3\beta^2\psi - 4\,\Gamma^3\beta\,\psi^2 + 2\,\Gamma^2\alpha\,\beta^2\psi \right. \\
&+ 4\,\Gamma^2\alpha\,\beta\,\psi^2 - \Gamma^2\alpha\,\psi^3 - 2\,\Gamma^2\beta^3\psi - 9\,\Gamma^2\beta\,\psi^3 + \Gamma^2\psi^4 + \Gamma\alpha\,\beta^2\psi^2 + 2\,\Gamma\alpha\,\beta\,\psi^3 - 2\,\Gamma\alpha\,\psi^4 \\
&\left. -\Gamma\beta^3\psi^2 - \Gamma\beta^2\psi^3 - 6\,\Gamma\beta\,\psi^4 + 2\,\Gamma\psi^5 - \alpha\,\psi^5 - \beta\,\psi^5 + \psi^6\right] \\
p_{11} &= \frac{128\,\beta^2\lambda}{D_2}\left[2\,\Gamma^4\beta + 2\,\Gamma^3\beta\,\alpha + \Gamma^3\alpha\,\psi - 4\,\Gamma^3\beta^2 + 3\,\Gamma^3\beta\,\psi - \Gamma^3\psi^2 - \Gamma^2\beta^2\alpha + 4\,\Gamma^2\beta\,\psi\,\alpha \right. \\
&+ 2\,\Gamma^2\psi^2\alpha + \Gamma^2\beta^3 - 9\,\Gamma^2\beta^2\psi - 2\,\Gamma^2\psi^3 - 2\,\Gamma\beta^2\psi\,\alpha + 2\,\Gamma\beta\,\psi^2\alpha + \Gamma\psi^3\alpha + 2\,\Gamma\beta^3\psi - 6\,\Gamma\beta^2\psi^2 \\
&\left. -\Gamma\beta\,\psi^3 - \Gamma\psi^4 - \beta^2\psi^2\alpha + \beta^3\psi^2 - \beta^2\psi^3\right] \\
p_{12} &= \frac{-64\,\beta\,\eta\,\lambda\,x}{D_2}\left[4\,\Gamma^4\beta^2 + 3\,\Gamma^3\alpha\,\beta^2 + 3\,\Gamma^3\alpha\,\beta\,\psi - 5\,\Gamma^3\beta^3 + 6\,\Gamma^3\beta^2\psi - 5\,\Gamma^3\beta\,\psi^2 - \Gamma^2\alpha\,\beta^3 \right. \\
&+ 6\,\Gamma^2\alpha\,\beta^2\psi + 6\,\Gamma^2\alpha\,\beta\,\psi^2 - \Gamma^2\alpha\,\psi^3 + \Gamma^2\beta^4 - 11\,\Gamma^2\beta^3\psi - 11\,\Gamma^2\beta\,\psi^3 + \Gamma^2\psi^4 - 2\,\Gamma\alpha\,\beta^3\psi \\
&+ 3\,\Gamma\alpha\,\beta^2\psi^2 + 3\,\Gamma\alpha\,\beta\,\psi^3 - 2\,\Gamma\alpha\,\psi^4 + 2\,\Gamma\beta^4\psi - 7\,\Gamma\beta^3\psi^2 - 2\,\Gamma\beta^2\psi^3 - 7\,\Gamma\beta\,\psi^4 \\
&\left. +2\,\Gamma\psi^5 - \alpha\,\beta^3\psi^2 - \alpha\,\psi^5 + \beta^4\psi^2 - \beta^3\psi^3 - \beta\,\psi^5 + \psi^6\right]
\end{aligned}
\tag{34}
$$

The second row coefficients, $p_{13},\ldots,p_{22}$ are given by,

$$
\begin{aligned}
p_{13} = {}& \frac{-128\,\beta^2\eta}{D_2}\,\big[\Gamma^4\alpha\,\beta - 5\,\Gamma^4\beta^2 - 3\,\Gamma^4\beta\,\psi - 3\,\Gamma^3\alpha\,\beta^2 - \Gamma^3\alpha\,\psi^2 + 5\,\Gamma^3\beta^3 - 9\,\Gamma^3\beta^2\psi - 5\,\Gamma^3\beta\,\psi^2 \\
& + \Gamma^3\psi^3 + \Gamma^2\alpha\,\beta^3 - 6\,\Gamma^2\alpha\,\beta^2\psi - 3\,\Gamma^2\alpha\,\beta\,\psi^2 - 2\,\Gamma^2\alpha\,\psi^3 - \Gamma^2\beta^4 + 11\,\Gamma^2\beta^3\psi - 3\,\Gamma^2\beta^2\psi^2 \\
& - \Gamma^2\beta\,\psi^3 + 2\,\Gamma^2\psi^4 + 2\,\Gamma\alpha\,\beta^3\psi - 3\,\Gamma\alpha\,\beta^2\psi^2 - 2\,\Gamma\alpha\,\beta\,\psi^3 - \Gamma\alpha\,\psi^4 - 2\,\Gamma\beta^4\psi \\
& + 7\,\Gamma\beta^3\psi^2 + \Gamma\beta^2\psi^3 + \Gamma\beta\,\psi^4 + \Gamma\psi^5 + \alpha\,\beta^3\psi^2 - \beta^4\psi^2 + \beta^3\psi^3\big] \\
p_{14} = {}& \frac{128\,\beta^2\lambda}{D_2}\,\big[\Gamma^4\alpha\,\beta - \Gamma^4\beta^2 - 3\,\Gamma^4\beta\,\psi + 3\,\Gamma^3\alpha\,\beta\,\psi - \Gamma^3\alpha\,\psi^2 - 3\,\Gamma^3\beta^2\psi - 10\,\Gamma^3\beta\,\psi^2 + \Gamma^3\psi^3 \\
& + 3\,\Gamma^2\alpha\,\beta\,\psi^2 - 3\,\Gamma^2\alpha\,\psi^3 - 3\,\Gamma^2\beta^2\psi^2 - 12\,\Gamma^2\beta\,\psi^3 + 3\,\Gamma^2\psi^4 + \Gamma\alpha\,\beta\,\psi^3 - 3\,\Gamma\alpha\,\psi^4 \\
& - \Gamma\beta^2\psi^3 - 6\,\Gamma\beta\,\psi^4 + 3\,\Gamma\psi^5 - \alpha\,\psi^5 - \beta\,\psi^5 + \psi^6\big] \\
p_{15} = {}& \frac{-32\,\Gamma\beta^2}{D_1}\,\big[\Gamma^2\beta\,\alpha - 3\,\Gamma^2\beta^2 - 3\,\Gamma^2\beta\,\psi - \alpha\,\Gamma\beta^2 - \Gamma\psi^2\alpha \\
& + \Gamma\beta^3 - 3\,\Gamma\beta^2\psi - 3\,\beta\,\Gamma\psi^2 + \Gamma\psi^3 - \alpha\,\beta^2\psi - \alpha\,\beta\,\psi^2 - \alpha\,\psi^3 + \beta^3\psi + \psi^4\big] \\
p_{16} = {}& \frac{-32\,\Gamma\beta^3}{D_1}\,\big[\Gamma^2\alpha - \Gamma^2\beta - \Gamma^2\psi + 2\,\Gamma\alpha\,\psi - 2\,\Gamma\beta\,\psi - 2\,\Gamma\psi^2 + \alpha\,\psi^2 - \beta\,\psi^2 - \psi^3\big] \\
p_{17} = {}& \frac{32\,\Gamma\beta^2}{D_1}\,\big[2\,\Gamma^2\beta\,\alpha - 4\,\Gamma^2\beta^2 - 4\,\Gamma^2\beta\,\psi - \alpha\,\Gamma\beta^2 + 2\,\alpha\,\Gamma\beta\,\psi - \Gamma\psi^2\alpha + \Gamma\beta^3 - 5\,\Gamma\beta^2\psi \\
& - 5\,\beta\,\Gamma\psi^2 + \Gamma\psi^3 - \alpha\,\beta^2\psi - \alpha\,\psi^3 + \beta^3\psi - \beta^2\psi^2 - \beta\,\psi^3 + \psi^4\big] \\
p_{18} = {}& \frac{-32\,\beta^2}{D_1}\,\big[2\,\Gamma^3\beta + 2\,\Gamma^2\beta\,\alpha + \Gamma^2\alpha\,\psi - 4\,\Gamma^2\beta^2 + \Gamma^2\beta\,\psi - \Gamma^2\psi^2 - \alpha\,\Gamma\beta^2 + 2\,\alpha\,\Gamma\beta\,\psi + \Gamma\psi^2\alpha \\
& + \Gamma\beta^3 - 5\,\Gamma\beta^2\psi - \beta\,\Gamma\psi^2 - \Gamma\psi^3 - \alpha\,\beta^2\psi + \beta^3\psi - \beta^2\psi^2\big] \\
p_{19} = {}& \frac{32\,\beta^2}{D_1}\,\big[2\,\Gamma^3\beta + \Gamma^2\alpha\,\psi + 5\,\Gamma^2\beta\,\psi - \Gamma^2\psi^2 + 2\,\Gamma\psi^2\alpha + 4\,\beta\,\Gamma\psi^2 - 2\,\Gamma\psi^3 + \alpha\,\psi^3 + \beta\,\psi^3 - \psi^4\big] \\
p_{20} = {}& \frac{32\,\beta^2}{D_1}\,\big[2\,\Gamma^2\beta\,\alpha - 4\,\Gamma^2\beta^2 - 4\,\Gamma^2\beta\,\psi - \alpha\,\Gamma\beta^2 + 2\,\alpha\,\Gamma\beta\,\psi - \Gamma\psi^2\alpha + \Gamma\beta^3 - 5\,\Gamma\beta^2\psi - 5\,\beta\,\Gamma\psi^2 \\
& + \Gamma\psi^3 - \alpha\,\beta^2\psi - \alpha\,\psi^3 + \beta^3\psi - \beta^2\psi^2 - \beta\,\psi^3 + \psi^4\big] \\
p_{21} = {}& \frac{128\,\Gamma\beta^2\eta}{D_2}\,\big[\Gamma^3\beta\,\alpha - 3\,\Gamma^3\beta^2 - 3\,\Gamma^3\beta\,\psi - \Gamma^2\beta^2\alpha + \Gamma^2\beta\,\psi\,\alpha - \Gamma^2\psi^2\alpha + \Gamma^2\beta^3 - 6\,\Gamma^2\beta^2\psi \\
& - 6\,\Gamma^2\beta\,\psi^2 + \Gamma^2\psi^3 - 2\,\Gamma\beta^2\psi\,\alpha - \Gamma\beta\,\psi^2\alpha - 2\,\Gamma\psi^3\alpha + 2\,\Gamma\beta^3\psi - 3\,\Gamma\beta^2\psi^2 - 3\,\Gamma\beta\,\psi^3 + 2\,\Gamma\psi^4 \\
& - \beta^2\psi^2\alpha - \alpha\,\beta\,\psi^3 - \psi^4\alpha + \beta^3\psi^2 + \psi^5\big] \\
p_{22} = {}& \frac{-128\,\Gamma\beta^2\lambda}{D_2}\,\big[\Gamma^3\beta\,\alpha - 3\,\Gamma^3\beta^2 - 3\,\Gamma^3\beta\,\psi - \Gamma^2\beta^2\alpha + \Gamma^2\beta\,\psi\,\alpha - \Gamma^2\psi^2\alpha + \Gamma^2\beta^3 - 6\,\Gamma^2\beta^2\psi \\
& - 6\,\Gamma^2\beta\,\psi^2 + \Gamma^2\psi^3 - 2\,\Gamma\beta^2\psi\,\alpha - \Gamma\beta\,\psi^2\alpha - 2\,\Gamma\psi^3\alpha + 2\,\Gamma\beta^3\psi - 3\,\Gamma\beta^2\psi^2 - 3\,\Gamma\beta\,\psi^3 \\
& + 2\,\Gamma\psi^4 - \beta^2\psi^2\alpha - \alpha\,\beta\,\psi^3 - \psi^4\alpha + \beta^3\psi^2 + \psi^5\big]
\end{aligned}
\tag{35}
$$

The third row coefficients, $p_{23},\ldots,p_{32}$ are given by,

$$
\begin{aligned}
p_{23} &= \frac{32\,\beta}{D_1}\,\big[\Gamma^3\beta\,\alpha - 3\,\Gamma^3\beta^2 - 3\,\Gamma^3\beta\,\psi - \Gamma^2\beta^2\alpha + \Gamma^2\beta\,\psi\,\alpha - \Gamma^2\psi^2\alpha + \Gamma^2\beta^3 - 6\,\Gamma^2\beta^2\psi \\
&\quad - 6\,\Gamma^2\beta\,\psi^2 + \Gamma^2\psi^3 - 2\,\Gamma\beta^2\psi\,\alpha - \Gamma\beta\,\psi^2\alpha - 2\,\Gamma\psi^3\alpha + 2\,\Gamma\beta^3\psi - 3\,\Gamma\beta^2\psi^2 - 3\,\Gamma\beta\,\psi^3 + 2\,\Gamma\psi^4 \\
&\quad -\beta^2\psi^2\alpha - \alpha\,\beta\,\psi^3 - \psi^4\alpha + \beta^3\psi^2 + \psi^5\big] \\
p_{24} &= \frac{32\,\beta^2}{D_1}\,\big[\Gamma^3\alpha - \Gamma^3\beta - \Gamma^3\psi + 3\,\Gamma^2\alpha\,\psi - 3\,\Gamma^2\beta\,\psi - 3\,\Gamma^2\psi^2 \\
&\quad +3\,\Gamma\psi^2\alpha - 3\,\beta\,\Gamma\psi^2 - 3\,\Gamma\psi^3 + \alpha\,\psi^3 - \beta\,\psi^3 - \psi^4\big] \\
p_{25} &= \frac{-32\,\beta}{D_1}\,\big[2\,\Gamma^3\beta\,\alpha - 4\,\Gamma^3\beta^2 - 4\,\Gamma^3\beta\,\psi - \Gamma^2\beta^2\alpha + 4\,\Gamma^2\beta\,\psi\,\alpha - \Gamma^2\psi^2\alpha + \Gamma^2\beta^3 - 9\,\Gamma^2\beta^2\psi \\
&\quad - 9\,\Gamma^2\beta\,\psi^2 + \Gamma^2\psi^3 - 2\,\Gamma\beta^2\psi\,\alpha + 2\,\Gamma\beta\,\psi^2\alpha - 2\,\Gamma\psi^3\alpha + 2\,\Gamma\beta^3\psi - 6\,\Gamma\beta^2\psi^2 - 6\,\Gamma\beta\,\psi^3 + 2\,\Gamma\psi^4 \\
&\quad -\beta^2\psi^2\alpha - \psi^4\alpha + \beta^3\psi^2 - \beta^2\psi^3 - \beta\,\psi^4 + \psi^5\big] \\
p_{26} &= \frac{128\,\beta^2\eta}{D_2}\,\big[2\,\Gamma^4\beta^2 + \Gamma^3\alpha\,\beta^2 + 2\,\Gamma^3\alpha\,\beta\,\psi - \Gamma^3\beta^3 + 3\,\Gamma^3\beta^2\psi - 4\,\Gamma^3\beta\,\psi^2 + 2\,\Gamma^2\alpha\,\beta^2\psi \\
&\quad + 4\,\Gamma^2\alpha\,\beta\,\psi^2 - \Gamma^2\alpha\,\psi^3 - 2\,\Gamma^2\beta^3\psi - 9\,\Gamma^2\beta\,\psi^3 + \Gamma^2\psi^4 + \Gamma\alpha\,\beta^2\psi^2 + 2\,\Gamma\alpha\,\beta\,\psi^3 \\
&\quad -2\,\Gamma\alpha\,\psi^4 - \Gamma\beta^3\psi^2 - \Gamma\beta^2\psi^3 - 6\,\Gamma\beta\,\psi^4 + 2\,\Gamma\psi^5 - \alpha\,\psi^5 - \beta\,\psi^5 + \psi^6\big] \\
p_{27} &= \frac{128\,\beta^3\lambda}{D_2}\,\big[2\,\Gamma^4\beta + 2\,\Gamma^3\beta\,\alpha + \Gamma^3\alpha\,\psi - 4\,\Gamma^3\beta^2 + 3\,\Gamma^3\beta\,\psi - \Gamma^3\psi^2 - \Gamma^2\beta^2\alpha + 4\,\Gamma^2\beta\,\psi\,\alpha \\
&\quad + 2\,\Gamma^2\psi^2\alpha + \Gamma^2\beta^3 - 9\,\Gamma^2\beta^2\psi - 2\,\Gamma^2\psi^3 - 2\,\Gamma\beta^2\psi\,\alpha + 2\,\Gamma\beta\,\psi^2\alpha + \Gamma\psi^3\alpha + 2\,\Gamma\beta^3\psi \\
&\quad -6\,\Gamma\beta^2\psi^2 - \Gamma\beta\,\psi^3 - \Gamma\psi^4 - \beta^2\psi^2\alpha + \beta^3\psi^2 - \beta^2\psi^3\big] \\
p_{28} &= \frac{-128\,\beta^2\eta}{D_2}\,\big[\Gamma^3\beta\,\alpha - 3\,\Gamma^3\beta^2 - 3\,\Gamma^3\beta\,\psi - \Gamma^2\beta^2\alpha + \Gamma^2\beta\,\psi\,\alpha - \Gamma^2\psi^2\alpha + \Gamma^2\beta^3 - 6\,\Gamma^2\beta^2\psi \\
&\quad - 6\,\Gamma^2\beta\,\psi^2 + \Gamma^2\psi^3 - 2\,\Gamma\beta^2\psi\,\alpha - \Gamma\beta\,\psi^2\alpha - 2\,\Gamma\psi^3\alpha + 2\,\Gamma\beta^3\psi - 3\,\Gamma\beta^2\psi^2 - 3\,\Gamma\beta\,\psi^3 \\
&\quad +2\,\Gamma\psi^4 - \beta^2\psi^2\alpha - \alpha\,\beta\,\psi^3 - \psi^4\alpha + \beta^3\psi^2 + \psi^5\big] \\
p_{29} &= \frac{128\,\beta^2\lambda}{D_2}\,\big[\Gamma^3\beta\,\alpha - 3\,\Gamma^3\beta^2 - 3\,\Gamma^3\beta\,\psi - \Gamma^2\beta^2\alpha + \Gamma^2\beta\,\psi\,\alpha - \Gamma^2\psi^2\alpha + \Gamma^2\beta^3 - 6\,\Gamma^2\beta^2\psi \\
&\quad - 6\,\Gamma^2\beta\,\psi^2 + \Gamma^2\psi^3 - 2\,\Gamma\beta^2\psi\,\alpha - \Gamma\beta\,\psi^2\alpha - 2\,\Gamma\psi^3\alpha + 2\,\Gamma\beta^3\psi - 3\,\Gamma\beta^2\psi^2 - 3\,\Gamma\beta\,\psi^3 \\
&\quad +2\,\Gamma\psi^4 - \beta^2\psi^2\alpha - \alpha\,\beta\,\psi^3 - \psi^4\alpha + \beta^3\psi^2 + \psi^5\big] \\
p_{30} &= \frac{-32\,\beta}{D_1}\,\big[\Gamma^3\beta\,\alpha - \Gamma^3\beta^2 - 3\,\Gamma^3\beta\,\psi + 2\,\Gamma^2\beta\,\psi\,\alpha - \Gamma^2\psi^2\alpha - 2\,\Gamma^2\beta^2\psi - 7\,\Gamma^2\beta\,\psi^2 + \Gamma^2\psi^3 \\
&\quad +\Gamma\beta\,\psi^2\alpha - 2\,\Gamma\psi^3\alpha - \Gamma\beta^2\psi^2 - 5\,\Gamma\beta\,\psi^3 + 2\,\Gamma\psi^4 - \psi^4\alpha - \beta\,\psi^4 + \psi^5\big] \\
p_{31} &= \frac{-32\,\beta^2}{D_1}\,\big[\Gamma^3\alpha - 3\,\Gamma^3\beta - \Gamma^3\psi - \Gamma^2\beta\,\alpha + 2\,\Gamma^2\alpha\,\psi + \Gamma^2\beta^2 - 7\,\Gamma^2\beta\,\psi - 2\,\Gamma^2\psi^2 - 2\,\alpha\,\Gamma\beta\,\psi \\
&\quad +\Gamma\psi^2\alpha + 2\,\Gamma\beta^2\psi - 5\,\beta\,\Gamma\psi^2 - \Gamma\psi^3 - \alpha\,\beta\,\psi^2 + \beta^2\psi^2 - \beta\,\psi^3\big] \\
p_{32} &= \frac{32\,\beta}{D_1}\,\big[2\,\Gamma^3\beta\,\alpha - 4\,\Gamma^3\beta^2 - 4\,\Gamma^3\beta\,\psi - \Gamma^2\beta^2\alpha + 4\,\Gamma^2\beta\,\psi\,\alpha - \Gamma^2\psi^2\alpha + \Gamma^2\beta^3 - 9\,\Gamma^2\beta^2\psi \\
&\quad - 9\,\Gamma^2\beta\,\psi^2 + \Gamma^2\psi^3 - 2\,\Gamma\beta^2\psi\,\alpha + 2\,\Gamma\beta\,\psi^2\alpha - 2\,\Gamma\psi^3\alpha + 2\,\Gamma\beta^3\psi - 6\,\Gamma\beta^2\psi^2 - 6\,\Gamma\beta\,\psi^3 \\
&\quad +2\,\Gamma\psi^4 - \beta^2\psi^2\alpha - \psi^4\alpha + \beta^3\psi^2 - \beta^2\psi^3 - \beta\,\psi^4 + \psi^5\big]
\end{aligned}
\tag{36}
$$

The fourth row coefficients, $p_{33}, \ldots, p_{40}$ are given by,

$$
\begin{aligned}
p_{33} &= \frac{-32\,\beta^2}{D_1}\left[2\,\Gamma^4\beta + 2\,\Gamma^3\beta\,\alpha + \Gamma^3\alpha\,\psi - 4\,\Gamma^3\beta^2 + 3\,\Gamma^3\beta\,\psi - \Gamma^3\psi^2 - \Gamma^2\beta^2\alpha + 4\,\Gamma^2\beta\,\psi\,\alpha + 2\,\Gamma^2\psi^2\alpha\right.\\
&\quad + \Gamma^2\beta^3 - 9\,\Gamma^2\beta^2\psi - 2\,\Gamma^2\psi^3 - 2\,\Gamma\beta^2\psi\,\alpha + 2\,\Gamma\beta\,\psi^2\alpha + \Gamma\psi^3\alpha + 2\,\Gamma\beta^3\psi - 6\,\Gamma\beta^2\psi^2\\
&\quad \left.-\Gamma\beta\,\psi^3 - \Gamma\psi^4 - \beta^2\psi^2\alpha + \beta^3\psi^2 - \beta^2\psi^3\right]\\
p_{34} &= \frac{32\,\beta^2}{D_1}\left[2\,\Gamma^4\beta + \Gamma^3\alpha\,\psi + 7\,\Gamma^3\beta\,\psi - \Gamma^3\psi^2 + 3\,\Gamma^2\psi^2\alpha + 9\,\Gamma^2\beta\,\psi^2\right.\\
&\quad \left.-3\,\Gamma^2\psi^3 + 3\,\Gamma\psi^3\alpha + 5\,\Gamma\beta\,\psi^3 - 3\,\Gamma\psi^4 + \psi^4\alpha + \beta\,\psi^4 - \psi^5\right]\\
p_{35} &= \frac{128\,\Gamma\beta^3\eta}{D_2}\left[\Gamma^3\beta\,\alpha - 3\,\Gamma^3\beta^2 - 3\,\Gamma^3\beta\,\psi - \Gamma^2\beta^2\alpha + \Gamma^2\beta\,\psi\,\alpha - \Gamma^2\psi^2\alpha + \Gamma^2\beta^3 - 6\,\Gamma^2\beta^2\psi\right.\\
&\quad - 6\,\Gamma^2\beta\,\psi^2 + \Gamma^2\psi^3 - 2\,\Gamma\beta^2\psi\,\alpha - \Gamma\beta\,\psi^2\alpha - 2\,\Gamma\psi^3\alpha + 2\,\Gamma\beta^3\psi - 3\,\Gamma\beta^2\psi^2 - 3\,\Gamma\beta\,\psi^3\\
&\quad \left.+2\,\Gamma\psi^4 - \beta^2\psi^2\alpha - \alpha\,\beta\,\psi^3 - \psi^4\alpha + \beta^3\psi^2 + \psi^5\right]\\
p_{36} &= \frac{-128\,\Gamma\beta^3\lambda}{D_2}\left[\Gamma^3\beta\,\alpha - 3\,\Gamma^3\beta^2 - 3\,\Gamma^3\beta\,\psi - \Gamma^2\beta^2\alpha + \Gamma^2\beta\,\psi\,\alpha - \Gamma^2\psi^2\alpha + \Gamma^2\beta^3\right.\\
&\quad - 6\,\Gamma^2\beta^2\psi - 6\,\Gamma^2\beta\,\psi^2 + \Gamma^2\psi^3 - 2\,\Gamma\beta^2\psi\,\alpha - \Gamma\beta\,\psi^2\alpha - 2\,\Gamma\psi^3\alpha + 2\,\Gamma\beta^3\psi - 3\,\Gamma\beta^2\psi^2 - 3\,\Gamma\beta\,\psi^3\\
&\quad \left.+2\,\Gamma\psi^4 - \beta^2\psi^2\alpha - \alpha\,\beta\,\psi^3 - \psi^4\alpha + \beta^3\psi^2 + \psi^5\right]\\
p_{37} &= \frac{128\,\beta^3\eta}{D_2}\left[2\,\Gamma^4\beta + 2\,\Gamma^3\beta\,\alpha + \Gamma^3\alpha\,\psi - 4\,\Gamma^3\beta^2 + 3\,\Gamma^3\beta\,\psi - \Gamma^3\psi^2 - \Gamma^2\beta^2\alpha + 4\,\Gamma^2\beta\,\psi\,\alpha\right.\\
&\quad + 2\,\Gamma^2\psi^2\alpha + \Gamma^2\beta^3 - 9\,\Gamma^2\beta^2\psi - 2\,\Gamma^2\psi^3 - 2\,\Gamma\beta^2\psi\,\alpha + 2\,\Gamma\beta\,\psi^2\alpha + \Gamma\psi^3\alpha + 2\,\Gamma\beta^3\psi - 6\,\Gamma\beta^2\psi^2\\
&\quad \left.-\Gamma\beta\,\psi^3 - \Gamma\psi^4 - \beta^2\psi^2\alpha + \beta^3\psi^2 - \beta^2\psi^3\right]\\
p_{38} &= \frac{128\,\beta^2\lambda}{D_2}\left(2\,\Gamma^4\beta^2 + \Gamma^3\alpha\,\beta^2 + 2\,\Gamma^3\alpha\,\beta\,\psi - \Gamma^3\beta^3 + 3\,\Gamma^3\beta^2\psi - 4\,\Gamma^3\beta\,\psi^2 + 2\,\Gamma^2\alpha\,\beta^2\psi\right.\\
&\quad + 4\,\Gamma^2\alpha\,\beta\,\psi^2 - \Gamma^2\alpha\,\psi^3 - 2\,\Gamma^2\beta^3\psi - 9\,\Gamma^2\beta\,\psi^3 + \Gamma^2\psi^4 + \Gamma\alpha\,\beta^2\psi^2 + 2\,\Gamma\alpha\,\beta\,\psi^3\\
&\quad \left.-2\,\Gamma\alpha\,\psi^4 - \Gamma\beta^3\psi^2 - \Gamma\beta^2\psi^3 - 6\,\Gamma\beta\,\psi^4 + 2\,\Gamma\psi^5 - \alpha\,\psi^5 - \beta\,\psi^5 + \psi^6\right]\\
p_{39} &= \frac{32\,\Gamma\beta^2}{D_1}\left[2\,\Gamma^3\beta + \Gamma^2\beta\,\alpha + \Gamma^2\alpha\,\psi - \Gamma^2\beta^2 + 4\,\Gamma^2\beta\,\psi - \Gamma^2\psi^2\right.\\
&\quad \left.+2\,\alpha\,\Gamma\beta\,\psi + 2\,\Gamma\psi^2\alpha - 2\,\Gamma\beta^2\psi + 2\,\beta\,\Gamma\psi^2 - 2\,\Gamma\psi^3 + \alpha\,\beta\,\psi^2 + \alpha\,\psi^3 - \beta^2\psi^2 - \psi^4\right]\\
p_{40} &= \frac{-32\,\Gamma\beta^2}{D_1}\left[2\,\Gamma^3\beta + \Gamma^2\beta\,\alpha + \Gamma^2\alpha\,\psi - \Gamma^2\beta^2 + 4\,\Gamma^2\beta\,\psi - \Gamma^2\psi^2 + 2\,\alpha\,\Gamma\beta\,\psi + 2\,\Gamma\psi^2\alpha - 2\,\Gamma\beta^2\psi\right.\\
&\quad \left.+2\,\beta\,\Gamma\psi^2 - 2\,\Gamma\psi^3 + \alpha\,\beta\,\psi^2 + \alpha\,\psi^3 - \beta^2\psi^2 - \psi^4\right]
\end{aligned}
\tag{37}
$$

The fifth row coefficients, $p_{41}, \ldots, p_{48}$ are given by,

$$
\begin{aligned}
p_{41} = & \frac{-128\,\beta^2\eta}{D_2}\,\big[\Gamma^4\alpha\,\beta - 3\,\Gamma^4\beta^2 - 3\,\Gamma^4\beta\,\psi - \Gamma^3\alpha\,\beta^2 + 2\,\Gamma^3\alpha\,\beta\,\psi - \Gamma^3\alpha\,\psi^2 + \Gamma^3\beta^3 - 9\,\Gamma^3\beta^2\psi \\
& - 9\,\Gamma^3\beta\,\psi^2 + \Gamma^3\psi^3 - 3\,\Gamma^2\alpha\,\beta^2\psi - 3\,\Gamma^2\alpha\,\psi^3 + 3\,\Gamma^2\beta^3\psi - 9\,\Gamma^2\beta^2\psi^2 - 9\,\Gamma^2\beta\,\psi^3 \\
& + 3\,\Gamma^2\psi^4 - 3\,\Gamma\alpha\,\beta^2\psi^2 - 2\,\Gamma\alpha\,\beta\,\psi^3 - 3\,\Gamma\alpha\,\psi^4 + 3\,\Gamma\beta^3\psi^2 - 3\,\Gamma\beta^2\psi^3 - 3\,\Gamma\beta\,\psi^4 + 3\,\Gamma\psi^5 \\
& -\alpha\,\beta^2\psi^3 - \alpha\,\beta\,\psi^4 - \alpha\,\psi^5 + \beta^3\psi^3 + \psi^6\big] \\
p_{42} = & \frac{128\,\beta^2\lambda}{D_2}\,\big[\Gamma^4\alpha\,\beta - 3\,\Gamma^4\beta^2 - 3\,\Gamma^4\beta\,\psi - \Gamma^3\alpha\,\beta^2 + 2\,\Gamma^3\alpha\,\beta\,\psi - \Gamma^3\alpha\,\psi^2 + \Gamma^3\beta^3 \\
& - 9\,\Gamma^3\beta^2\psi - 9\,\Gamma^3\beta\,\psi^2 + \Gamma^3\psi^3 - 3\,\Gamma^2\alpha\,\beta^2\psi - 3\,\Gamma^2\alpha\,\psi^3 + 3\,\Gamma^2\beta^3\psi - 9\,\Gamma^2\beta^2\psi^2 - 9\,\Gamma^2\beta\,\psi^3 \\
& + 3\,\Gamma^2\psi^4 - 3\,\Gamma\alpha\,\beta^2\psi^2 - 2\,\Gamma\alpha\,\beta\,\psi^3 - 3\,\Gamma\alpha\,\psi^4 + 3\,\Gamma\beta^3\psi^2 - 3\,\Gamma\beta^2\psi^3 - 3\,\Gamma\beta\,\psi^4 \\
& +3\,\Gamma\psi^5 - \alpha\,\beta^2\psi^3 - \alpha\,\beta\,\psi^4 - \alpha\,\psi^5 + \beta^3\psi^3 + \psi^6\big] \\
p_{43} = & \frac{-32\,\beta^2}{D_1}\,\big(\Gamma^3\beta\,\alpha - \Gamma^3\beta^2 - 3\,\Gamma^3\beta\,\psi + 2\,\Gamma^2\beta\,\psi\,\alpha - \Gamma^2\psi^2\alpha - 2\,\Gamma^2\beta^2\psi - 7\,\Gamma^2\beta\,\psi^2 + \Gamma^2\psi^3 \\
& +\Gamma\beta\,\psi^2\alpha - 2\,\Gamma\psi^3\alpha - \Gamma\beta^2\psi^2 - 5\,\Gamma\beta\,\psi^3 + 2\,\Gamma\psi^4 - \psi^4\alpha - \beta\,\psi^4 + \psi^5\big] \\
p_{44} = & \frac{-32\,\beta^3}{D_1}\,\big[\Gamma^3\alpha - 3\,\Gamma^3\beta - \Gamma^3\psi - \Gamma^2\beta\,\alpha + 2\,\Gamma^2\alpha\,\psi + \Gamma^2\beta^2 - 7\,\Gamma^2\beta\,\psi - 2\,\Gamma^2\psi^2 - 2\,\alpha\,\Gamma\beta\,\psi \\
& +\Gamma\psi^2\alpha + 2\,\Gamma\beta^2\psi - 5\,\beta\,\Gamma\psi^2 - \Gamma\psi^3 - \alpha\,\beta\,\psi^2 + \beta^2\psi^2 - \beta\,\psi^3\big] \\
p_{45} = & \frac{-32\,\beta^2}{D_1}\,\big[2\,\Gamma^3\beta + \Gamma^2\beta\,\alpha + \Gamma^2\alpha\,\psi - \Gamma^2\beta^2 + 4\,\Gamma^2\beta\,\psi - \Gamma^2\psi^2 + 2\,\alpha\,\Gamma\beta\,\psi + 2\,\Gamma\psi^2\alpha - 2\,\Gamma\beta^2\psi \\
& +2\,\beta\,\Gamma\psi^2 - 2\,\Gamma\psi^3 + \alpha\,\beta\,\psi^2 + \alpha\,\psi^3 - \beta^2\psi^2 - \psi^4\big] \\
p_{46} = & \frac{32\,\beta^2}{D_1}\,\big[2\,\Gamma^3\beta + \Gamma^2\beta\,\alpha + \Gamma^2\alpha\,\psi - \Gamma^2\beta^2 + 4\,\Gamma^2\beta\,\psi - \Gamma^2\psi^2 + 2\,\alpha\,\Gamma\beta\,\psi + 2\,\Gamma\psi^2\alpha - 2\,\Gamma\beta^2\psi \\
& +2\,\beta\,\Gamma\psi^2 - 2\,\Gamma\psi^3 + \alpha\,\beta\,\psi^2 + \alpha\,\psi^3 - \beta^2\psi^2 - \psi^4\big] \\
p_{47} = & \frac{128\,\beta^2\eta}{D_2}\,\big[\Gamma^4\alpha\,\beta - \Gamma^4\beta^2 - 3\,\Gamma^4\beta\,\psi + 3\,\Gamma^3\alpha\,\beta\,\psi - \Gamma^3\alpha\,\psi^2 - 3\,\Gamma^3\beta^2\psi - 10\,\Gamma^3\beta\,\psi^2 \\
& + \Gamma^3\psi^3 + 3\,\Gamma^2\alpha\,\beta\,\psi^2 - 3\,\Gamma^2\alpha\,\psi^3 - 3\,\Gamma^2\beta^2\psi^2 - 12\,\Gamma^2\beta\,\psi^3 + 3\,\Gamma^2\psi^4 + \Gamma\alpha\,\beta\,\psi^3 - 3\,\Gamma\alpha\,\psi^4 \\
& -\Gamma\beta^2\psi^3 - 6\,\Gamma\beta\,\psi^4 + 3\,\Gamma\psi^5 - \alpha\,\psi^5 - \beta\,\psi^5 + \psi^6\big] \\
p_{48} = & \frac{-128\,\beta^2\lambda}{D_2}\,\big[\Gamma^4\alpha\,\beta - 5\,\Gamma^4\beta^2 - 3\,\Gamma^4\beta\,\psi - 3\,\Gamma^3\alpha\,\beta^2 - \Gamma^3\alpha\,\psi^2 + 5\,\Gamma^3\beta^3 - 9\,\Gamma^3\beta^2\psi \\
& - 5\,\Gamma^3\beta\,\psi^2 + \Gamma^3\psi^3 + \Gamma^2\alpha\,\beta^3 - 6\,\Gamma^2\alpha\,\beta^2\psi - 3\,\Gamma^2\alpha\,\beta\,\psi^2 - 2\,\Gamma^2\alpha\,\psi^3 - \Gamma^2\beta^4 + 11\,\Gamma^2\beta^3\psi \\
& - 3\,\Gamma^2\beta^2\psi^2 - \Gamma^2\beta\,\psi^3 + 2\,\Gamma^2\psi^4 + 2\,\Gamma\alpha\,\beta^3\psi - 3\,\Gamma\alpha\,\beta^2\psi^2 - 2\,\Gamma\alpha\,\beta\,\psi^3 - \Gamma\alpha\,\psi^4 - 2\,\Gamma\beta^4\psi \\
& +7\,\Gamma\beta^3\psi^2 + \Gamma\beta^2\psi^3 + \Gamma\beta\,\psi^4 + \Gamma\psi^5 + \alpha\,\beta^3\psi^2 - \beta^4\psi^2 + \beta^3\psi^3\big]
\end{aligned}
\tag{38}
$$

The sixth row coefficients, $p_{49},\dots,p_{56}$ are given as,

$$
\begin{aligned}
p_{49} = {} & \frac{128\,\beta^3\eta}{D_2}\,\big[2\,\Gamma^5\beta + 2\,\Gamma^4\alpha\,\beta + \Gamma^4\alpha\,\psi - 4\,\Gamma^4\beta^2 + 5\,\Gamma^4\beta\,\psi - \Gamma^4\psi^2 - \Gamma^3\alpha\,\beta^2 + 6\,\Gamma^3\alpha\,\beta\,\psi \\
& + 3\,\Gamma^3\alpha\,\psi^2 + \Gamma^3\beta^3 - 13\,\Gamma^3\beta^2\psi + 3\,\Gamma^3\beta\,\psi^2 - 3\,\Gamma^3\psi^3 - 3\,\Gamma^2\alpha\,\beta^2\psi + 6\,\Gamma^2\alpha\,\beta\,\psi^2 + 3\,\Gamma^2\alpha\,\psi^3 \\
& + 3\,\Gamma^2\beta^3\psi - 15\,\Gamma^2\beta^2\psi^2 - \Gamma^2\beta\,\psi^3 - 3\,\Gamma^2\psi^4 - 3\,\Gamma\alpha\,\beta^2\psi^2 + 2\,\Gamma\alpha\,\beta\,\psi^3 + \Gamma\alpha\,\psi^4 + 3\,\Gamma\beta^3\psi^2 \\
& -7\,\Gamma\beta^2\psi^3 - \Gamma\beta\,\psi^4 - \Gamma\psi^5 - \alpha\,\beta^2\psi^3 + \beta^3\psi^3 - \beta^2\psi^4\big] \\
p_{50} = {} & \frac{128\,\beta^2\lambda}{D_2}\,\big[2\,\Gamma^5\beta^2 + \Gamma^4\alpha\,\beta^2 + 2\,\Gamma^4\alpha\,\beta\,\psi - \Gamma^4\beta^3 + 5\,\Gamma^4\beta^2\psi - 4\,\Gamma^4\beta\,\psi^2 + 3\,\Gamma^3\alpha\,\beta^2\psi \\
& + 6\,\Gamma^3\alpha\,\beta\,\psi^2 - \Gamma^3\alpha\,\psi^3 - 3\,\Gamma^3\beta^3\psi + 3\,\Gamma^3\beta^2\psi^2 - 13\,\Gamma^3\beta\,\psi^3 + \Gamma^3\psi^4 + 3\,\Gamma^2\alpha\,\beta^2\psi^2 \\
& + 6\,\Gamma^2\alpha\,\beta\,\psi^3 - 3\,\Gamma^2\alpha\,\psi^4 - 3\,\Gamma^2\beta^3\psi^2 - \Gamma^2\beta^2\psi^3 - 15\,\Gamma^2\beta\,\psi^4 + 3\,\Gamma^2\psi^5 + \Gamma\alpha\,\beta^2\psi^3 \\
& +2\,\Gamma\alpha\,\beta\,\psi^4 - 3\,\Gamma\alpha\,\psi^5 - \Gamma\beta^3\psi^3 - \Gamma\beta^2\psi^4 - 7\,\Gamma\beta\,\psi^5 + 3\,\Gamma\psi^6 - \alpha\,\psi^6 - \beta\,\psi^6 + \psi^7\big] \\
p_{51} = {} & \frac{32\,\Gamma\beta^3}{D_1}\,\big[2\,\Gamma^3\beta + \Gamma^2\beta\,\alpha + \Gamma^2\alpha\,\psi - \Gamma^2\beta^2 + 4\,\Gamma^2\beta\,\psi - \Gamma^2\psi^2 + 2\,\alpha\,\Gamma\beta\,\psi + 2\,\Gamma\psi^2\alpha - 2\,\Gamma\beta^2\psi \\
& +2\,\beta\,\Gamma\psi^2 - 2\,\Gamma\psi^3 + \alpha\,\beta\,\psi^2 + \alpha\,\psi^3 - \beta^2\psi^2 - \psi^4\big] \\
p_{52} = {} & \frac{-32\,\Gamma\beta^3}{D_1}\,\big[2\,\Gamma^3\beta + \Gamma^2\beta\,\alpha + \Gamma^2\alpha\,\psi - \Gamma^2\beta^2 + 4\,\Gamma^2\beta\,\psi - \Gamma^2\psi^2 + 2\,\alpha\,\Gamma\beta\,\psi + 2\,\Gamma\psi^2\alpha - 2\,\Gamma\beta^2\psi \\
& +2\,\beta\,\Gamma\psi^2 - 2\,\Gamma\psi^3 + \alpha\,\beta\,\psi^2 + \alpha\,\psi^3 - \beta^2\psi^2 - \psi^4\big] \\
p_{53} = {} & \frac{-32\,\beta^3}{D_1}\,\big[\Gamma^3\alpha - 3\,\Gamma^3\beta - \Gamma^3\psi - \Gamma^2\beta\,\alpha + 2\,\Gamma^2\alpha\,\psi + \Gamma^2\beta^2 - 7\,\Gamma^2\beta\,\psi - 2\,\Gamma^2\psi^2 - 2\,\alpha\,\Gamma\beta\,\psi \\
& +\Gamma\psi^2\alpha + 2\,\Gamma\beta^2\psi - 5\,\beta\,\Gamma\psi^2 - \Gamma\psi^3 - \alpha\,\beta\,\psi^2 + \beta^2\psi^2 - \beta\,\psi^3\big] \\
p_{54} = {} & \frac{-32\,\beta^2}{D_1}\,\big[\Gamma^3\beta\,\alpha - \Gamma^3\beta^2 - 3\,\Gamma^3\beta\,\psi + 2\,\Gamma^2\beta\,\psi\,\alpha - \Gamma^2\psi^2\alpha - 2\,\Gamma^2\beta^2\psi - 7\,\Gamma^2\beta\,\psi^2 + \Gamma^2\psi^3 \\
& +\Gamma\beta\,\psi^2\alpha - 2\,\Gamma\psi^3\alpha - \Gamma\beta^2\psi^2 - 5\,\Gamma\beta\,\psi^3 + 2\,\Gamma\psi^4 - \psi^4\alpha - \beta\,\psi^4 + \psi^5\big] \\
p_{55} = {} & \frac{-128\,\Gamma\beta^3\eta}{D_2}\,\big[2\,\Gamma^4\beta + \Gamma^3\beta\,\alpha + \Gamma^3\alpha\,\psi - \Gamma^3\beta^2 + 6\,\Gamma^3\beta\,\psi - \Gamma^3\psi^2 + 3\,\Gamma^2\beta\,\psi\,\alpha + 3\,\Gamma^2\psi^2\alpha \\
& - 3\,\Gamma^2\beta^2\psi + 6\,\Gamma^2\beta\,\psi^2 - 3\,\Gamma^2\psi^3 + 3\,\Gamma\beta\,\psi^2\alpha + 3\,\Gamma\psi^3\alpha - 3\,\Gamma\beta^2\psi^2 + 2\,\Gamma\beta\,\psi^3 - 3\,\Gamma\psi^4 \\
& +\alpha\,\beta\,\psi^3 + \psi^4\alpha - \beta^2\psi^3 - \psi^5\big] \\
p_{56} = {} & \frac{-128\,\Gamma\beta^2\lambda}{D_2}\,\big[2\,\Gamma^4\beta^2 + 2\,\Gamma^3\alpha\,\beta^2 + 2\,\Gamma^3\alpha\,\beta\,\psi - 4\,\Gamma^3\beta^3 - 4\,\Gamma^3\beta\,\psi^2 - \Gamma^2\alpha\,\beta^3 + 3\,\Gamma^2\alpha\,\beta^2\psi \\
& + 3\,\Gamma^2\alpha\,\beta\,\psi^2 - \Gamma^2\alpha\,\psi^3 + \Gamma^2\beta^4 - 8\,\Gamma^2\beta^3\psi - 6\,\Gamma^2\beta^2\psi^2 - 8\,\Gamma^2\beta\,\psi^3 + \Gamma^2\psi^4 \\
& - 2\,\Gamma\alpha\,\beta^3\psi - 2\,\Gamma\alpha\,\psi^4 + 2\,\Gamma\beta^4\psi - 4\,\Gamma\beta^3\psi^2 - 4\,\Gamma\beta^2\psi^3 - 4\,\Gamma\beta\,\psi^4 + 2\,\Gamma\psi^5 - \alpha\,\beta^3\psi^2 \\
& -\alpha\,\beta^2\psi^3 - \alpha\,\beta\,\psi^4 - \alpha\,\psi^5 + \beta^4\psi^2 + \psi^6\big]
\end{aligned}
\tag{39}
$$

Finally, the seventh row coefficients, $p_{57},\ldots,p_{64}$ are

$$\begin{aligned}
p_{57} &= \frac{-32\,\beta^2}{D_1}\left[2\,\Gamma^4\beta+\Gamma^3\beta\,\alpha+\Gamma^3\alpha\,\psi-\Gamma^3\beta^2+6\,\Gamma^3\beta\,\psi-\Gamma^3\psi^2+3\,\Gamma^2\beta\,\psi\,\alpha+3\,\Gamma^2\psi^2\alpha-3\,\Gamma^2\beta^2\psi\right.\\
&\quad+6\,\Gamma^2\beta\,\psi^2-3\,\Gamma^2\psi^3+3\,\Gamma\beta\,\psi^2\alpha+3\,\Gamma\psi^3\alpha-3\,\Gamma\beta^2\psi^2+2\,\Gamma\beta\,\psi^3-3\,\Gamma\psi^4+\alpha\,\beta\,\psi^3\\
&\quad\left.+\psi^4\alpha-\beta^2\psi^3-\psi^5\right]\\
p_{58} &= \frac{32\,\beta^2}{D_1}\left[2\,\Gamma^4\beta+\Gamma^3\beta\,\alpha+\Gamma^3\alpha\,\psi-\Gamma^3\beta^2+6\,\Gamma^3\beta\,\psi-\Gamma^3\psi^2+3\,\Gamma^2\beta\,\psi\,\alpha+3\,\Gamma^2\psi^2\alpha-3\,\Gamma^2\beta^2\psi\right.\\
&\quad+6\,\Gamma^2\beta\,\psi^2-3\,\Gamma^2\psi^3+3\,\Gamma\beta\,\psi^2\alpha+3\,\Gamma\psi^3\alpha-3\,\Gamma\beta^2\psi^2+2\,\Gamma\beta\,\psi^3-3\,\Gamma\psi^4\\
&\quad\left.+\alpha\,\beta\,\psi^3+\psi^4\alpha-\beta^2\psi^3-\psi^5\right]\\
p_{59} &= \frac{128\,\beta^3\eta}{D_2}\left[\Gamma^4\alpha\,\beta-\Gamma^4\beta^2-3\,\Gamma^4\beta\,\psi+3\,\Gamma^3\alpha\,\beta\,\psi-\Gamma^3\alpha\,\psi^2-3\,\Gamma^3\beta^2\psi-10\,\Gamma^3\beta\,\psi^2+\Gamma^3\psi^3\right.\\
&\quad+3\,\Gamma^2\alpha\,\beta\,\psi^2-3\,\Gamma^2\alpha\,\psi^3-3\,\Gamma^2\beta^2\psi^2-12\,\Gamma^2\beta\,\psi^3+3\,\Gamma^2\psi^4+\Gamma\alpha\,\beta\,\psi^3-3\,\Gamma\alpha\,\psi^4\\
&\quad\left.-\Gamma\beta^2\psi^3-6\,\Gamma\beta\,\psi^4+3\,\Gamma\psi^5-\alpha\,\psi^5-\beta\,\psi^5+\psi^6\right]\\
p_{60} &= \frac{-128\,\beta^3\lambda}{D_2}\left[\Gamma^4\alpha\,\beta-5\,\Gamma^4\beta^2-3\,\Gamma^4\beta\,\psi-3\,\Gamma^3\alpha\,\beta^2-\Gamma^3\alpha\,\psi^2+5\,\Gamma^3\beta^3-9\,\Gamma^3\beta^2\psi-5\,\Gamma^3\beta\,\psi^2\right.\\
&\quad+\Gamma^3\psi^3+\Gamma^2\alpha\,\beta^3-6\,\Gamma^2\alpha\,\beta^2\psi-3\,\Gamma^2\alpha\,\beta\,\psi^2-2\,\Gamma^2\alpha\,\psi^3-\Gamma^2\beta^4+11\,\Gamma^2\beta^3\psi-3\,\Gamma^2\beta^2\psi^2\\
&\quad-\Gamma^2\beta\,\psi^3+2\,\Gamma^2\psi^4+2\,\Gamma\alpha\,\beta^3\psi-3\,\Gamma\alpha\,\beta^2\psi^2-2\,\Gamma\alpha\,\beta\,\psi^3-\Gamma\alpha\,\psi^4-2\,\Gamma\beta^4\psi\\
&\quad\left.+7\,\Gamma\beta^3\psi^2+\Gamma\beta^2\psi^3+\Gamma\beta\,\psi^4+\Gamma\psi^5+\alpha\,\beta^3\psi^2-\beta^4\psi^2+\beta^3\psi^3\right] \qquad (40)\\
p_{61} &= \frac{128\,\beta^3\eta}{D_2}\left[2\,\Gamma^4\beta+\Gamma^3\beta\,\alpha+\Gamma^3\alpha\,\psi-\Gamma^3\beta^2+6\,\Gamma^3\beta\,\psi-\Gamma^3\psi^2+3\,\Gamma^2\beta\,\psi\,\alpha+3\,\Gamma^2\psi^2\alpha-3\,\Gamma^2\beta^2\psi\right.\\
&\quad+6\,\Gamma^2\beta\,\psi^2-3\,\Gamma^2\psi^3+3\,\Gamma\beta\,\psi^2\alpha+3\,\Gamma\psi^3\alpha-3\,\Gamma\beta^2\psi^2\\
&\quad\left.+2\,\Gamma\beta\,\psi^3-3\,\Gamma\psi^4+\alpha\,\beta\,\psi^3+\psi^4\alpha-\beta^2\psi^3-\psi^5\right]\\
p_{62} &= \frac{128\,\beta^2\lambda}{D_2}\left[2\,\Gamma^4\beta^2+2\,\Gamma^3\alpha\,\beta^2+2\,\Gamma^3\alpha\,\beta\,\psi-4\,\Gamma^3\beta^3-4\,\Gamma^3\beta\,\psi^2-\Gamma^2\alpha\,\beta^3+3\,\Gamma^2\alpha\,\beta^2\psi\right.\\
&\quad+3\,\Gamma^2\alpha\,\beta\,\psi^2-\Gamma^2\alpha\,\psi^3+\Gamma^2\beta^4-8\,\Gamma^2\beta^3\psi-6\,\Gamma^2\beta^2\psi^2-8\,\Gamma^2\beta\,\psi^3+\Gamma^2\psi^4\\
&\quad-2\,\Gamma\alpha\,\beta^3\psi-2\,\Gamma\alpha\,\psi^4+2\,\Gamma\beta^4\psi-4\,\Gamma\beta^3\psi^2-4\,\Gamma\beta^2\psi^3-4\,\Gamma\beta\,\psi^4+2\,\Gamma\psi^5-\alpha\,\beta^3\psi^2-\alpha\,\beta^2\psi^3\\
&\quad\left.-\alpha\,\beta\,\psi^4-\alpha\,\psi^5+\beta^4\psi^2+\psi^6\right]\\
p_{63} &= \frac{32\,\beta^2}{D_1}\left[2\,\Gamma^4\beta+\Gamma^3\alpha\,\psi+7\,\Gamma^3\beta\,\psi-\Gamma^3\psi^2+3\,\Gamma^2\psi^2\alpha+9\,\Gamma^2\beta\,\psi^2-3\,\Gamma^2\psi^3\right.\\
&\quad\left.+3\,\Gamma\psi^3\alpha+5\,\Gamma\beta\,\psi^3-3\,\Gamma\psi^4+\psi^4\alpha+\beta\,\psi^4-\psi^5\right]\\
p_{64} &= \frac{-32\,\beta^2}{D_1}\left[2\,\Gamma^4\beta+2\,\Gamma^3\beta\,\alpha+\Gamma^3\alpha\,\psi-4\,\Gamma^3\beta^2+3\,\Gamma^3\beta\,\psi-\Gamma^3\psi^2-\Gamma^2\beta^2\alpha+4\,\Gamma^2\beta\,\psi\,\alpha+2\,\Gamma^2\psi^2\alpha\right.\\
&\quad+\Gamma^2\beta^3-9\,\Gamma^2\beta^2\psi-2\,\Gamma^2\psi^3-2\,\Gamma\beta^2\psi\,\alpha+2\,\Gamma\beta\,\psi^2\alpha+\Gamma\psi^3\alpha+2\,\Gamma\beta^3\psi-6\,\Gamma\beta^2\psi^2\\
&\quad\left.-\Gamma\beta\,\psi^3-\Gamma\psi^4-\beta^2\psi^2\alpha+\beta^3\psi^2-\beta^2\psi^3\right]
\end{aligned}$$

It must be noted that these expressions can be simplified further by exploring the recurrence of terms in the various coefficients. For example, there is a clear repetition in the coefficients of the diagonal terms that can be combined and reduce the total number of the presented coefficients.

References

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