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A HIGH ORDER ANALYTIC CONTINUATION TECHNIQUE FOR THE PERTURBED TWO-BODY PROBLEM STATE TRANSITION MATRIX

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In this work, the analytic continuation technique is used to derive the State Transition Matrix for the perturbed two body problem resulting in a fast, high precision solution that outperforms state of the art numerical methods. Analytic Continuation is a Taylor series based technique where two scalar Lagrange-like invariants (f = **r.r** and $g_p = f^{-p/2}$) are defined and differentiated to an arbitrary order by using Leibniz product rule. These derivatives are used in a Taylor series expansion to obtain the solution. Previously, this method has been applied to several trajectory calculations, that resulted in high precision solutions for both position and velocity with a comparatively lower computational cost. As future work, the method will be expanded to solve the perturbed multi revolution Lambert's problem.

INTRODUCTION

The fast-growing space industry is focusing on the robotic and nanosatellites to autonomously rendezvous and dock with other satellites to carry out complex jobs like refueling and repairing.[1] It is not possible to carry out these tasks without high precision in error propagation of the initial state vector. Autonomous satellite navigation also requires frequent update of the state vector and the covariance matrix. For an instance, GPSPAC navigation system processes Global Positioning System (GPS) data every 6 seconds.[2,3] However, nanosatellites work on low power budget, which requires an orbit determination algorithm that can run on processors that consume comparatively low power with high precision results.[4] Lambert's problem is another fundamental problem in Astrodynamics, which solves for targeting spacecraft and missiles. It is dependent on the sensitivity towards the initial state vector and the State Transition Matrix (STM) can be utilized to solve the problem. The computation cost for the STM is comparatively higher than other components of the whole orbit determination procedure because of the cumbersome computation of the Jacobian Matrix.[5]

The STM in astrodynamics describes how errors in initial position and velocity propagate over time. It works as a sensitivity matrix. Hence, it can also be used in control theory to calculate and minimize the errors of the initial states.[6] There already exists an exact analytical representation of the two-body cartesian STM presented by Goodyear.[7] The process is relatively easy for computation and valid for all types of orbits for the attractive force, and for the hyperbolic and rectilinear

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orbits for the repulsive force.[7] However, it requires the evaluation of transcendental functions. Additionally, it is not easy to introduce third body perturbation in this method.[3] Kuga simplified this method to increase the numerical efficiency for the Keplerian elliptical orbit.[5,8] However, this simplified method does not include the J_2 perturbation term.[5] Markley presented an approximation to the cartesian STM.[3] Although J_2 perturbation can be included in Markley's method and the method can be applied into the n-body problem, it is restricted by short time intervals. This method can include the J_2 perturbation term only up to 6 seconds time step.[3] Julie Read, et al. applied Modified Chebyshev Picard Iteration to formulate STM with Spherical Harmonic Gravity model, which is well suited for parallel implementation for additional speed of computaiton.[15] Recently Hatten and Russel developed a decoupled direct method using fixed step size Runge-Kutta method for first and second order STM, which decouples the state propagation and the solution of the STMs to make the calculation computationally efficient.[14]

Yamanaka and Ankersen linearized the differential equations of relative motion on elliptical orbits and derived a convenient STM for practical engineering application, which is valid for all unperturbed elliptical orbits.[16] Gim and Alfriend developed geometric method to derive the STM for relative motion with gravitational perturbation, which uses the relationship between the relative states and differential orbital elements.[17] Koenig, et al. used classical Keplerian orbit elements to derive the STM and incorporated J_2 and differential drag.[18]

The Analytic Continuation method is a numerical integration method based on Taylor series expansion and Leibniz product rule. Previously this method has been proved very precise in solving several two body unperturbed trajectories.[9] Later on, J_2-J_6 gravity perturbation terms were introduced in the approach.[10] Work had also been done to introduce atmospheric drag in two body problem with this method.[11] In this paper, the Analytic Continuation Method is used to derive the State Transition Matrix for the perturbed two-body problem. As an initial step, results for the unperturbed case have been compared with Lagrange's F & G solution. The method is then expanded for the J_2 perturbation term and compared against ODE45 (based on Runge Kutta (4,5) integration method[13]). The results show the symplectic check and provide a high precision solution for the newly derived STMs.

ANALYTIC CONTINUATION FOR DERIVING THE STM

The Analytic Continuation technique relies on the definition of two Lagrange-like invariants as, $f = \mathbf{r}.\mathbf{r}$ and $g_p = f^{-p/2}$ where $\mathbf{r}$ is the position vector. For the state variables (position and velocity) propagation for the two-body problem, the acceleration vector is given by,

$$\boldsymbol{r}^{(2)}(t) = -\mu \frac{\boldsymbol{r}(t)}{\left(\boldsymbol{r}(t).\boldsymbol{r}(t)\right)^{3/2}} \tag{1}$$

where, $\mathbf{r}(t)$ is the current position vector and μ is the standard gravitational parameter.

Leibnitz product rule is then used to compute higher order derivatives of the variables via the recursive formulas:

$$\boldsymbol{r}^{(n+2)}(t) = -\mu \sum_{m=0}^{n} \binom{n}{m} \boldsymbol{r}^{(m)}(t) g_3^{(n-m)}(t) \tag{2}$$

$$f^{(n)}(t) = \sum_{m=0}^{n} \binom{n}{m} \boldsymbol{r}^{(m)}(t).\boldsymbol{r}^{(n-m)}(t) \tag{3}$$

$$g_3^{(n+1)}(t) = -\frac{1}{f(t)}\left\{\frac{3}{2}f^{(1)}(t)g_3^{(n)}(t) + \sum_{m=1}^{n}\binom{n}{m}\left(\frac{3}{2}f^{(m+1)}(t)g_3^{(n-m)}(t) + f^{(m)}(t)g_3^{(n-m+1)}(t)\right)\right\} \tag{4}$$

Finally, the higher order derivatives of the acceleration are substituted into the Taylor series expansion to obtain position and velocity as shown in Eq. (5) and Eq. (6),

$$\boldsymbol{r}(t+dT) = \boldsymbol{r}(t) + \sum_{m=1}^{n} \boldsymbol{r}^{(m)}(t)\frac{dT^{(m)}}{m!} \tag{5}$$

$$\boldsymbol{r}^{(1)}(t+dT) = \boldsymbol{r}^{(1)}(t) + \sum_{m=2}^{n} \boldsymbol{r}^{(m)}(t)\frac{dT^{(m-1)}}{(m-1)!} \tag{6}$$

The STM for the two-body problem is defined as, [6,12]

$$\phi = \begin{bmatrix} \phi_{11}(t+dT,t) & \phi_{12}(t+dT,t) \\ \phi_{21}(t+dT,t) & \phi_{22}(t+dT,t) \end{bmatrix} = \begin{bmatrix} \frac{\partial \boldsymbol{r}(t+dT)}{\partial \boldsymbol{r}(t)} & \frac{\partial \boldsymbol{r}(t+dT)}{\partial \boldsymbol{r}^{(1)}(t)} \\ \frac{\partial \boldsymbol{r}^{(1)}(t+dT)}{\partial \boldsymbol{r}(t)} & \frac{\partial \boldsymbol{r}^{(1)}(t+dT)}{\partial \boldsymbol{r}^{(1)}(t)} \end{bmatrix} \tag{7}$$

where, Φ_{11} is the sensitivity of the next position to the current position, Φ_{12} is the sensitivity of the next position to the current velocity, Φ_{21} is the sensitivity of the next velocity to the current position and Φ_{22} is the sensitivity of the next velocity to the current velocity.

Following the same Taylor series expansion of the state variables the elements of the STM can be obtained from

$$\phi_{11}(t+dT,t) = \frac{\partial \boldsymbol{r}(t+dT)}{\partial \boldsymbol{r}(t)} = \frac{\partial \boldsymbol{r}(t)}{\partial \boldsymbol{r}(t)} + \sum_{m=1}^{n} \frac{\partial \boldsymbol{r}^{(m)}(t)}{\partial \boldsymbol{r}(t)}\frac{dT^{(m)}}{m!} \tag{8}$$

$$\phi_{12}(t+dT,t) = \frac{\partial \boldsymbol{r}(t+dT)}{\partial \boldsymbol{r}^{(1)}(t)} = \frac{\partial \boldsymbol{r}(t)}{\partial \boldsymbol{r}^{(1)}(t)} + \sum_{m=1}^{n} \frac{\partial \boldsymbol{r}^{(m)}(t)}{\partial \boldsymbol{r}^{(1)}(t)}\frac{dT^{(m)}}{m!} \tag{9}$$

$$\phi_{21}(t+dT,t) = \frac{\partial \boldsymbol{r}^{(1)}(t+dT)}{\partial \boldsymbol{r}(t)} = \frac{\partial \boldsymbol{r}^{(1)}(t)}{\partial \boldsymbol{r}(t)} + \sum_{m=2}^{n} \frac{\partial \boldsymbol{r}^{(m)}(t)}{\partial \boldsymbol{r}(t)}\frac{dT^{(m-1)}}{(m-1)!} \tag{10}$$

$$\phi_{22}(t+dT,t) = \frac{\partial \boldsymbol{r}^{(1)}(t+dT)}{\partial \boldsymbol{r}^{(1)}(t)} = \frac{\partial \boldsymbol{r}^{(1)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} + \sum_{m=2}^{n} \frac{\partial \boldsymbol{r}^{(m)}(t)}{\partial \boldsymbol{r}^{(1)}(t)}\frac{dT^{(m-1)}}{(m-1)!} \tag{11}$$

where, dT is the time step between the current position and the next position.

The partial derivatives of **r**(t) and **r** $^{(1)}$(t) with respect to **r**(t) and **r** $^{(1)}$(t) can be written as,

$$\frac{\partial \boldsymbol{r}(t)}{\partial \boldsymbol{r}(t)} = \frac{\partial \boldsymbol{r}^{(1)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} = \boldsymbol{I}_{3\times3} \tag{12}$$

$$\frac{\partial \boldsymbol{r}(t)}{\partial \boldsymbol{r}^{(1)}(t)} = \frac{\partial \boldsymbol{r}^{(1)}(t)}{\partial \boldsymbol{r}(t)} = \boldsymbol{0}_{3\times3} \tag{13}$$

The partial derivatives of the higher order terms (up to sixth order) with respect to **r**(t) and **r**$^{(1)}$(t) are shown in the Appendix.

An approach is developed to calculate the series expansions of the 3×3 sub-matrices using recursive formulas, so that the series can be extended to any arbitrary number of partial derivatives of the higher order terms. For this reason, a new variable **F**, is defined as,

$$\boldsymbol{F} \equiv \boldsymbol{r}.\boldsymbol{r}^{\boldsymbol{T}} \tag{14}$$

The derived recursive formulas to calculate partial derivatives of the higher order terms with respect to position and velocity are given by,

$$\frac{\partial \boldsymbol{r}^{(n+2)}(t)}{\partial \boldsymbol{r}(t)} = \frac{\partial \boldsymbol{r}^{(n+3)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} = \mu\left\{3\sum_{m=0}^{n}\binom{n}{m} g_5^{(m)}(t)\boldsymbol{F}^{(n-m)} - I g_3^{(n)}(t)\right\} \tag{15}$$

It is worth noting that Eq. (15) shows a symmetry in the evaluation of the position and velocity partials using the derived recursion. The algorithm of the whole process is shown in Table 1 where, for brevity the notation **r** is used instead of **r**(t).

Table 1. Algorithm for Unperturbed STM Computation using Analytic Continuation Method

Initialize **r**(t_0), **r**(t_0)$^{(1)}$, dT

for k = 1 to i_{max}, where i_{max} is the total time step

$$\boldsymbol{r}^{(2)} = -\mu\frac{\boldsymbol{r}}{(\boldsymbol{r}.\boldsymbol{r})^{3/2}};$$

$$f = \boldsymbol{r}.\boldsymbol{r};$$

$$f^{(1)} = 2\boldsymbol{r}.\boldsymbol{r}^{(1)};$$

$$f^{(2)} = 2\boldsymbol{r}^{(1)}.\boldsymbol{r}^{(1)} + 2\boldsymbol{r}.\boldsymbol{r}^{(2)};$$

$$g_3 = f^{-\frac{3}{2}};$$

$$g_3^{(1)} = \left(-\frac{3}{2}\right) f^{(1)}\frac{g_3}{f};$$

$$g_3^{(2)} = \left(-\frac{1}{f}\right)\left(\frac{3}{2}f^{(1)}g_3^{(1)} + \frac{3}{2}f^{(2)}g_3 + f^{(1)}g_3^{(1)}\right);$$

$$g_5 = f^{-\frac{5}{2}};$$

$$g_5^{(1)} = \left(-\frac{5}{2}\right)f^{(1)}\frac{g_5}{f};$$

$$g_5^{(2)} = \left(-\frac{1}{f}\right)\left(\frac{5}{2}f^{(1)}g_5^{(1)} + \frac{5}{2}f^{(2)}g_5 + f^{(1)}g_5^{(1)}\right);$$

for n = 1 to N, where N is the total number of derivatives

$$\boldsymbol{r}^{(n+2)} = -\mu\sum_{m=0}^{n}\binom{n}{m}\boldsymbol{r}^{(m)}g_3^{(n-m)};$$

$$f^{(n)} = \sum_{m=0}^{n}\binom{n}{m}\boldsymbol{r}^{(m)}.\boldsymbol{r}^{(n-m)};$$

$$g_3^{(n+1)} = -\frac{1}{f}\left\{\frac{3}{2}f^{(1)}g_3^{(n)} + \sum_{m=1}^{n}\binom{n}{m}\left(\frac{3}{2}f^{(m+1)}g_3^{(n-m)} + f^{(m)}g_3^{(n-m+1)}\right)\right\};$$

$$g_5^{(n+1)} = -\frac{1}{f}\left\{\frac{5}{2}f^{(1)}g_5^{(n)} + \sum_{m=1}^{n}\binom{n}{m}\left(\frac{5}{2}f^{(m+1)}g_5^{(n-m)} + f^{(m)}g_5^{(n-m+1)}\right)\right\};$$

end for

$$\boldsymbol{r}(t+dT) = \boldsymbol{r}(t) + \sum_{m=1}^{n}\boldsymbol{r}^{(m)}(t)\frac{dT^{(m)}}{m!};$$

$$\boldsymbol{r}^{(1)}(t+dT) = \boldsymbol{r}^{(1)}(t) + \sum_{m=2}^{n}\boldsymbol{r}^{(m)}(t)\frac{dT^{(m-1)}}{(m-1)!};$$

$$\frac{\partial\boldsymbol{r}}{\partial\boldsymbol{r}} = \boldsymbol{I}_{3\times3};$$

$$\frac{\partial\boldsymbol{r}^{(1)}}{\partial\boldsymbol{r}^{(1)}} = \boldsymbol{I}_{3\times3};$$

$$\frac{\partial\boldsymbol{r}^{(1)}}{\partial\boldsymbol{r}} = \boldsymbol{0}_{3\times3};$$

$$\frac{\partial\boldsymbol{r}}{\partial\boldsymbol{r}^{(1)}} = \boldsymbol{0}_{3\times3};$$

for n = 1 to N

$$\frac{\partial\boldsymbol{r}^{(n+2)}}{\partial\boldsymbol{r}} = \mu\left\{3\sum_{m=0}^{n}\binom{n}{m}g_5^{(m)}\boldsymbol{F}^{(n-m)} - Ig_3^{(n)}\right\};$$

$$\frac{\partial \boldsymbol{r}^{(n+3)}}{\partial \boldsymbol{r}^{(1)}} = \frac{\partial \boldsymbol{r}^{(n+2)}}{\partial \boldsymbol{r}};$$

end for

$$\phi_{11}(t+dT,t) = \frac{\partial \boldsymbol{r}(t+dT)}{\partial \boldsymbol{r}(t)} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{r}} + \sum_{m=1}^{n} \frac{\partial \boldsymbol{r}^{(m)}}{\partial \boldsymbol{r}} \frac{dT^{(m)}}{m!} ;$$

$$\phi_{12}(t+dT,t) = \frac{\partial \boldsymbol{r}(t+dT)}{\partial \boldsymbol{r}^{(1)}(t)} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{r}^{(1)}} + \sum_{m=1}^{n} \frac{\partial \boldsymbol{r}^{(m)}}{\partial \boldsymbol{r}^{(1)}} \frac{dT^{(m)}}{m!};$$

$$\phi_{21}(t+dT,t) = \frac{\partial \boldsymbol{r}^{(1)}(t+dT)}{\partial \boldsymbol{r}(t)} = \frac{\partial \boldsymbol{r}^{(1)}}{\partial \boldsymbol{r}} + \sum_{m=2}^{n} \frac{\partial \boldsymbol{r}^{(m)}}{\partial \boldsymbol{r}} \frac{dT^{(m-1)}}{(m-1)!};$$

$$\phi_{22}(t+dT,t) = \frac{\partial \boldsymbol{r}^{(1)}(t+dT)}{\partial \boldsymbol{r}^{(1)}(t)} = \frac{\partial \boldsymbol{r}^{(1)}}{\partial \boldsymbol{r}^{(1)}} + \sum_{m=2}^{n} \frac{\partial \boldsymbol{r}^{(m)}}{\partial \boldsymbol{r}^{(1)}} \frac{dT^{(m-1)}}{(m-1)!};$$

end for

Next, J_2 perturbation is applied to the states and the State Transition Matrix. In this case, the J_2 perturbation acceleration is defined as,

$$a_{J_2} = -\frac{3}{2} J_2 \left(\frac{\mu}{r^2}\right)\left(\frac{r_{eq}}{r}\right)^2 \begin{pmatrix} \left(1-5\left(\frac{z}{r}\right)^2\right)\frac{x}{r} \\ \left(1-5\left(\frac{z}{r}\right)^2\right)\frac{y}{r} \\ \left(3-5\left(\frac{z}{r}\right)^2\right)\frac{z}{r} \end{pmatrix} = -\frac{3}{2} J_2 \mu {r_{eq}}^2 \begin{pmatrix} x g_5 - 5xz^2 g_7 \\ y g_5 - 5yz^2 g_7 \\ 3z g_5 - 5z^3 g_7 \end{pmatrix} \tag{16}$$

where, r_{eq} is equatorial radius of earth.

To add the J_2 perturbation to the higher order time derivatives of the position, the higher order time derivatives of the perturbation acceleration is calculated. For simplicity, when calculating the higher order time derivatives of the perturbed acceleration, 1 constant and 10 new variables are defined as,

$$\begin{aligned} &B_a = x g_5 \qquad B_b = y g_5 \qquad B_c = z g_5 \\ &B_d = z g_7 \qquad B_e = xz \qquad B_f = yz \\ &B_g = zz \qquad B_h = B_e B_d \qquad B_i = B_f B_d \\ &B_j = B_g B_d \qquad C_{J_2} = -\frac{3}{2} J_2 \mu r_{eq}^2 \end{aligned} \tag{17}$$

Now, the higher order time derivatives of the newly defined variables are given by,

$$
\begin{aligned}
{B_a}^{(n)} &= \sum_{m=0}^{n} \binom{n}{m} x^{(m)} {g_5}^{(n-m)} & {B_b}^{(n)} &= \sum_{m=0}^{n} \binom{n}{m} y^{(m)} {g_5}^{(n-m)} \\
{B_c}^{(n)} &= \sum_{m=0}^{n} \binom{n}{m} z^{(m)} {g_5}^{(n-m)} & {B_d}^{(n)} &= \sum_{m=0}^{n} \binom{n}{m} z^{(m)} {g_7}^{(n-m)} \\
{B_e}^{(n)} &= \sum_{m=0}^{n} \binom{n}{m} x^{(m)} z^{(n-m)} & {B_f}^{(n)} &= \sum_{m=0}^{n} \binom{n}{m} y^{(m)} z^{(n-m)} \\
{B_g}^{(n)} &= \sum_{m=0}^{n} \binom{n}{m} z^{(m)} z^{(n-m)} & {B_h}^{(n)} &= \sum_{m=0}^{n} \binom{n}{m} {B_e}^{(m)} {B_d}^{(n-m)} \\
{B_i}^{(n)} &= \sum_{m=0}^{n} \binom{n}{m} {B_f}^{(m)} {B_d}^{(n-m)} & {B_j}^{(n)} &= \sum_{m=0}^{n} \binom{n}{m} {B_g}^{(m)} {B_d}^{(n-m)}
\end{aligned}
\tag{18}
$$

Finally, using the higher order time derivatives of the newly defined variables, the higher order time derivatives of the J_2 perturbation acceleration is computed as,

$$
a_{J_2}^{(n)} = r_{J_2}^{(n+2)} = C_{J_2} \begin{pmatrix} {B_a}^{(n)} - 5{B_h}^{(n)} \\ {B_b}^{(n)} - 5{B_i}^{(n)} \\ 3{B_c}^{(n)} - 5{B_j}^{(n)} \end{pmatrix}
\tag{19}
$$

where, r_{J2} and $r_{J2}^{(1)}$ are position and velocity perturbation and equal to zero.

In order to derive the STM with J_2 perturbation, the newly computed derivations are added to the previous work as shown in the algorithm in Table 2.

Table 2. Algorithm for the Perturbed STM Computation using Analytic Continuation Method

Initialize $\mathbf{r}(t_0)$, $\mathbf{r}^{(1)}(t_0)$, dT

$$
a_{J_2} = -\frac{3}{2} J_2 \left(\frac{\mu}{r^2}\right)\left(\frac{r_{eq}}{r}\right)^2 \begin{pmatrix} \left(1 - 5\left(\frac{z}{r}\right)^2\right)\frac{x}{r} \\ \left(1 - 5\left(\frac{z}{r}\right)^2\right)\frac{y}{r} \\ \left(3 - 5\left(\frac{z}{r}\right)^2\right)\frac{z}{r} \end{pmatrix};
$$

$$
C_{J_2} = -\frac{3}{2} J_2 \mu r_{eq}^2 \,;
$$

for k = 1 to i_{max}, where i_{max} is the total time step

$$
\boldsymbol{r}^{(2)} = -\mu \frac{\boldsymbol{r}}{(\boldsymbol{r}.\boldsymbol{r})^{3/2}};
$$

$$
f = \boldsymbol{r}.\boldsymbol{r}\,;
$$

$$
f^{(1)} = 2\boldsymbol{r}.\boldsymbol{r}^{(1)}\,;
$$

$$f^{(2)} = 2\boldsymbol{r}^{(1)}.\boldsymbol{r}^{(1)} + 2\boldsymbol{r}.\boldsymbol{r}^{(2)} ;$$

$$g_3 = f^{-\frac{3}{2}} ;$$

$$g_3^{(1)} = \left(-\frac{3}{2}\right) f^{(1)} \frac{g_3}{f} ;$$

$$g_3^{(2)} = \left(-\frac{1}{f}\right)\left(\frac{3}{2} f^{(1)} g_3^{(1)} + \frac{3}{2} f^{(2)} g_3 + f^{(1)} g_3^{(1)}\right) ;$$

$$g_5 = f^{-\frac{5}{2}} ;$$

$$g_5^{(1)} = \left(-\frac{5}{2}\right) f^{(1)} \frac{g_5}{f} ;$$

$$g_5^{(2)} = \left(-\frac{1}{f}\right)\left(\frac{5}{2} f^{(1)} g_5^{(1)} + \frac{5}{2} f^{(2)} g_5 + f^{(1)} g_5^{(1)}\right) ;$$

$$g_7 = f^{-\frac{7}{2}} ;$$

$$g_7^{(1)} = \left(-\frac{7}{2}\right) f^{(1)} \frac{g_7}{f} ;$$

$$g_7^{(2)} = \left(-\frac{1}{f}\right)\left(\frac{7}{2} f^{(1)} g_7^{(1)} + \frac{7}{2} f^{(2)} g_7 + f^{(1)} g_7^{(1)}\right) ;$$

$$B_a = x g_5 ; \qquad B_a^{(1)} = x g_5^{(1)} + x^{(1)} g_5 ; \qquad B_a^{(2)} = x g_5^{(2)} + 2x^{(1)} g_5^{(1)} + x^{(2)} g_5 ;$$

$$B_b = y g_5 ; \qquad B_b^{(1)} = y g_5^{(1)} + y^{(1)} g_5 ; \qquad B_b^{(2)} = y g_5^{(2)} + 2y^{(1)} g_5^{(1)} + y^{(2)} g_5 ;$$

$$B_c = z g_5 ; \qquad B_c^{(1)} = z g_5^{(1)} + z^{(1)} g_5 ; \qquad B_c^{(2)} = z g_5^{(2)} + 2z^{(1)} g_5^{(1)} + z^{(2)} g_5 ;$$

$$B_d = z g_7 ; \qquad B_d^{(1)} = z g_7^{(1)} + z^{(1)} g_7 ; \qquad B_d^{(2)} = z g_7^{(2)} + 2z^{(1)} g_7^{(1)} + z^{(2)} g_7 ;$$

$$B_e = xz ; \qquad B_e^{(1)} = x z^{(1)} + x^{(1)} z ; \qquad B_e^{(2)} = x z^{(2)} + 2x^{(1)} z^{(1)} + x^{(2)} z ;$$

$$B_f = yz ; \qquad B_f^{(1)} = y z^{(1)} + y^{(1)} z ; \qquad B_f^{(2)} = y z^{(2)} + 2y^{(1)} z^{(1)} + y^{(2)} z ;$$

$$B_g = zz ; \qquad B_g^{(1)} = z z^{(1)} + z^{(1)} z ; \qquad B_g^{(2)} = z z^{(2)} + 2z^{(1)} z^{(1)} + z^{(2)} z ;$$

$$B_h = B_e B_d ; \qquad B_h^{(1)} = B_e B_d^{(1)} + B_e^{(1)} B_d ; \qquad B_h^{(2)} = B_e B_d^{(2)} + 2B_e^{(1)} B_d^{(1)} + B_e^{(2)} B_d ;$$

$$B_i = B_f B_d ; \qquad B_i^{(1)} = B_f B_d^{(1)} + B_f^{(1)} B_d ; \qquad B_i^{(2)} = B_f B_d^{(2)} + 2B_f^{(1)} B_d^{(1)} + B_f^{(2)} B_d ;$$

$$B_j = B_g B_d ; \qquad B_j^{(1)} = B_g B_d^{(1)} + B_g^{(1)} B_d ; \qquad B_j^{(2)} = B_g B_d^{(2)} + 2B_g^{(1)} B_d^{(1)} + B_g^{(2)} B_d ;$$

$$r_{J_2}^{(2)} = C_{J_2} \begin{bmatrix} B_a - 5B_h \\ B_b - 5B_i \\ 3B_c - 5B_j \end{bmatrix} ; \qquad r_{J_2}^{(3)} = C_{J_2} \begin{bmatrix} B_a^{(1)} - 5B_h^{(1)} \\ B_b^{(1)} - 5B_i^{(1)} \\ 3B_c^{(1)} - 5B_j^{(1)} \end{bmatrix} ; \qquad r_{J_2}^{(4)} = C_{J_2} \begin{bmatrix} B_a^{(2)} - 5B_h^{(2)} \\ B_b^{(2)} - 5B_i^{(2)} \\ 3B_c^{(2)} - 5B_j^{(2)} \end{bmatrix} ;$$

for n = 1 to N, where N is the total number of derivatives

$$\boldsymbol{r}^{(n+2)} = -\mu \sum_{m=0}^{n} \binom{n}{m} \boldsymbol{r}^{(m)} g_3^{(n-m)};$$

$$f^{(n)} = \sum_{m=0}^{n} \binom{n}{m} \boldsymbol{r}^{(m)} . \boldsymbol{r}^{(n-m)};$$

$$g_3^{(n+1)} = -\frac{1}{f} \left\{ \frac{3}{2} f^{(1)} g_3^{(n)} + \sum_{m=1}^{n} \binom{n}{m} \left(\frac{3}{2} f^{(m+1)} g_3^{(n-m)} + f^{(m)} g_3^{(n-m+1)} \right) \right\};$$

$$g_5^{(n+1)} = -\frac{1}{f} \left\{ \frac{5}{2} f^{(1)} g_5^{(n)} + \sum_{m=1}^{n} \binom{n}{m} \left(\frac{5}{2} f^{(m+1)} g_5^{(n-m)} + f^{(m)} g_5^{(n-m+1)} \right) \right\};$$

$$g_7^{(n+1)} = -\frac{1}{f} \left\{ \frac{7}{2} f^{(1)} g_7^{(n)} + \sum_{m=1}^{n} \binom{n}{m} \left(\frac{7}{2} f^{(m+1)} g_7^{(n-m)} + f^{(m)} g_7^{(n-m+1)} \right) \right\};$$

$$B_a{}^{(n)} = \sum_{m=0}^{n} \binom{n}{m} x^{(m)} g_5{}^{(n-m)}; \qquad B_b{}^{(n)} = \sum_{m=0}^{n} \binom{n}{m} y^{(m)} g_5{}^{(n-m)};$$

$$B_c{}^{(n)} = \sum_{m=0}^{n} \binom{n}{m} z^{(m)} g_5{}^{(n-m)}; \qquad B_d{}^{(n)} = \sum_{m=0}^{n} \binom{n}{m} z^{(m)} g_7{}^{(n-m)};$$

$$B_e{}^{(n)} = \sum_{m=0}^{n} \binom{n}{m} x^{(m)} z^{(n-m)}; \qquad B_f{}^{(n)} = \sum_{m=0}^{n} \binom{n}{m} y^{(m)} z^{(n-m)};$$

$$B_g{}^{(n)} = \sum_{m=0}^{n} \binom{n}{m} z^{(m)} z^{(n-m)}; \qquad B_h{}^{(n)} = \sum_{m=0}^{n} \binom{n}{m} B_e{}^{(m)} B_d{}^{(n-m)};$$

$$B_i{}^{(n)} = \sum_{m=0}^{n} \binom{n}{m} B_f{}^{(m)} B_d{}^{(n-m)}; \qquad B_j{}^{(n)} = \sum_{m=0}^{n} \binom{n}{m} B_g{}^{(m)} B_d{}^{(n-m)};$$

$$r_{J_2}^{(n+2)} = C_{J_2} \begin{pmatrix} B_a{}^{(n)} - 5B_h{}^{(n)} \\ B_b{}^{(n)} - 5B_i{}^{(n)} \\ 3B_c{}^{(n)} - 5B_j{}^{(n)} \end{pmatrix};$$

end for

$$\boldsymbol{r}(t + dT) = \boldsymbol{r}(t) + \sum_{m=1}^{n} \boldsymbol{r}^{(m)}(t) \frac{dT^{(m)}}{m!};$$

$$\boldsymbol{r}^{(1)}(t + dT) = \boldsymbol{r}^{(1)}(t) + \sum_{m=2}^{n} \boldsymbol{r}^{(m)}(t) \frac{dT^{(m-1)}}{(m-1)!};$$

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{r}} = \boldsymbol{I}_{3\times3};$$

$$\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{r}^{(1)}} = \boldsymbol{0}_{3\times3};$$

$$\frac{\partial \boldsymbol{r}^{(1)}}{\partial \boldsymbol{r}} = \boldsymbol{0}_{3\times3};$$

$$\frac{\partial \boldsymbol{r}^{(1)}}{\partial \boldsymbol{r}^{(1)}} = \boldsymbol{I}_{3\times 3}\,;$$

for n = 1 to N

$$\frac{\partial \boldsymbol{r}^{(n+2)}}{\partial \boldsymbol{r}} = \mu\left\{3\sum_{m=0}^{n}\binom{n}{m} g_5^{(m)}\boldsymbol{F}^{(n-m)} - I g_3^{(n)}\right\};$$

$$\frac{\partial \boldsymbol{r}^{(n+3)}}{\partial \boldsymbol{r}^{(1)}} = \frac{\partial \boldsymbol{r}^{(n+2)}}{\partial \boldsymbol{r}};$$

end for

$$\phi_{11}(t+dT,t) = \frac{\partial \boldsymbol{r}(t+dT)}{\partial \boldsymbol{r}(t)} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{r}} + \sum_{m=1}^{n}\frac{\partial \boldsymbol{r}^{(m)}}{\partial \boldsymbol{r}}\frac{dT^{(m)}}{m!}\,;$$

$$\phi_{12}(t+dT,t) = \frac{\partial \boldsymbol{r}(t+dT)}{\partial \boldsymbol{r}(t)^{(1)}} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{r}^{(1)}} + \sum_{m=1}^{n}\frac{\partial \boldsymbol{r}^{(m)}}{\partial \boldsymbol{r}^{(1)}}\frac{dT^{(m)}}{m!};$$

$$\phi_{21}(t+dT,t) = \frac{\partial \boldsymbol{r}(t+dT)^{(1)}}{\partial \boldsymbol{r}(t)} = \frac{\partial \boldsymbol{r}^{(1)}}{\partial \boldsymbol{r}} + \sum_{m=2}^{n}\frac{\partial \boldsymbol{r}^{(m)}}{\partial \boldsymbol{r}}\frac{dT^{(m-1)}}{(m-1)!};$$

$$\phi_{22}(t+dT,t) = \frac{\partial \boldsymbol{r}(t+dT)^{(1)}}{\partial \boldsymbol{r}(t)^{(1)}} = \frac{\partial \boldsymbol{r}^{(1)}}{\partial \boldsymbol{r}^{(1)}} + \sum_{m=2}^{n}\frac{\partial \boldsymbol{r}^{(m)}}{\partial \boldsymbol{r}^{(1)}}\frac{dT^{(m-1)}}{(m-1)!};$$

end for

NUMERICAL RESULTS

In this section numerical results are presented for the STM derived via the Analytical Continuation technique for 4 orbits as shown in Table 3. First, the new approach is utilized to compute the STM for the unperturbed orbits and compared versus the closed-form solution by Battin.[12] Finally, J_2 perturbation is added and the STM is computed and compared versus ODE45 in terms of accuracy. All codes are written and compiled using MATLAB R2016b.

Table 3. Orbits Used for Numerical Simulation

Orbit Type	a, m	e	f, deg	i, deg	ω, deg	Ω, deg	t_p, s
LEO	7.3090×10^6	0.1	0	60	30	45	6.2187×10^3
MEO	1.0964×10^7	0.4	0	60	30	45	1.1424×10^4
GTO	2.6352×10^7	0.6	0	60	30	45	4.2574×10^4
HEO	2.6999×10^7	0.7	0	60	30	45	4.4152×10^4

To compare the numerical results, three different methods are used; calculation of RMS error, energy conservation and the symplectic check.

The RMS Error of each element of the STM in the time domain of 10 orbit period is computed from difference between the elements of the unperturbed STMs computed using Battin's method[12] and the Analytic Continuation technique as shown in Eq. (20).

$$E_{ij} = \sqrt{\sum_{k=1}^{n}\left(M_{ijk} - L_{ijk}\right)^2/n} \tag{20}$$

where, M_{ijk} and L_{ijk} are the (i,j)th terms of the STMs at kth time period from Battin's method[12] and Analytic Continuation method respectively, E_{ij} is the RMS Error of the (i,j)th term of the STMs and n is the total number of STMs (time-steps).

The total energy is computed at each time step over 10 orbit periods, using Eq. (21) and compared against the initial total energy. The total energy check is computed as the normalized difference as shown in Eq. (22).

$$Total\ Energy_{(i)} = \frac{1}{2}\boldsymbol{r}_{(i)}^{(1)T}\boldsymbol{r}_{(i)}^{(1)} - \frac{\mu}{r_{(i)}} + \frac{1}{2}J_2\frac{\mu}{r_{(i)}}\left(\frac{r_{eq}}{r_{(i)}}\right)^2\left(3\left(\frac{z_{(i)}}{r_{(i)}}\right)^2 - 1\right) \tag{21}$$

$$Total\ Energy\ Check = \left|\frac{Total\ Energy_{(i)} - Total\ Energy}{Total\ Energy}\right| \tag{22}$$

where, Total Energy is the total energy at the initial conditions and Total $Energy_{(i)}$ is the total energy at the ith time step.

Finally, the symplectic nature of the perturbed STMs is examined and elements of error matrix are plotted. The matrix [Φ] is called symplectic, if it satisfies Eq. (23) [6]

$$[\phi]^{\boldsymbol{T}}[J][\phi] = [J] \tag{23}$$

where, [Φ] is the STM and [J] is a skew-symmetric matrix defined by Eq. (24)

$$[J] = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ -I_{3\times3} & 0_{3\times3} \end{bmatrix} \tag{24}$$

The error in the symplectic nature of the STMs are calculated by Eq. (25) [15]

$$\left[E_{sym.}\right] = [\phi]^{\boldsymbol{T}}[J][\phi] - [J] \tag{25}$$

where, $[E_{sym.}]$ is the symplectic error matrix.

The STMs are calculated in every 25s for 10 orbit periods for the unperturbed cases using Analytic Continuation method and the simplified F & G solution for the State Transition Matrices by Battin.[7] The RMS error for the 36 elements of the STMs are calculated using Eq. (20) for the four orbit test cases and the results are shown in Table 4 to Table 7.

Table 4. RMS Error of the Elements of The Unperturbed STM of LEO Orbit

	3.9241×10^{-8}	1.7154×10^{-8}	1.5946×10^{-8}	2.4383×10^{-5}	2.4725×10^{-5}	2.6308×10^{-5}	
Φ_{11}	1.7835×10^{-8}	4.1506×10^{-8}	1.6939×10^{-8}	2.4725×10^{-5}	2.4312×10^{-5}	2.6993×10^{-5}	Φ_{12}
	1.5464×10^{-8}	1.7660×10^{-8}	4.3214×10^{-8}	2.6309×10^{-5}	2.6993×10^{-5}	2.9202×10^{-5}	
	6.3071×10^{-9}	2.7438×10^{-9}	2.5737×10^{-9}	3.9189×10^{-6}	3.9742×10^{-6}	4.2288×10^{-6}	
Φ_{21}	2.8805×10^{-9}	6.6710×10^{-9}	2.7087×10^{-9}	3.9741×10^{-6}	3.9078×10^{-6}	4.3389×10^{-6}	Φ_{22}
	2.4768×10^{-9}	2.8535×10^{-9}	6.9455×10^{-9}	4.2288×10^{-6}	4.3390×10^{-6}	4.6945×10^{-6}	

Table 5. RMS Error of the Elements of The Unperturbed STM of MEO Orbit

	1.9383×10^{-8}	1.0136×10^{-8}	5.7495×10^{-9}	1.5192×10^{-5}	1.6535×10^{-5}	1.4957×10^{-5}	
Φ_{11}	1.0633×10^{-8}	2.5392×10^{-8}	9.9908×10^{-9}	1.6534×10^{-5}	1.4897×10^{-5}	1.7753×10^{-5}	Φ_{12}
	5.8882×10^{-9}	1.0568×10^{-8}	2.0691×10^{-8}	1.4957×10^{-5}	1.7754×10^{-5}	1.7968×10^{-5}	
	3.0528×10^{-9}	1.5868×10^{-9}	9.0400×10^{-10}	2.3916×10^{-6}	2.6043×10^{-6}	2.3560×10^{-6}	
Φ_{21}	1.6846×10^{-9}	3.9993×10^{-9}	1.5624×10^{-9}	2.6041×10^{-6}	2.3464×10^{-6}	2.7965×10^{-6}	Φ_{22}
	9.3131×10^{-10}	1.6761×10^{-9}	3.2588×10^{-9}	2.3560×10^{-6}	2.7968×10^{-6}	2.8318×10^{-6}	

Table 6. RMS Error of the Elements of The Unperturbed STM of GTO Orbit

	7.0046×10^{-10}	3.8750×10^{-10}	1.8786×10^{-10}	1.2381×10^{-6}	1.3800×10^{-6}	1.1720×10^{-6}	
Φ_{11}	3.9740×10^{-10}	9.7111×10^{-10}	3.8274×10^{-10}	1.3800×10^{-6}	1.2054×10^{-6}	1.4728×10^{-6}	Φ_{12}
	1.9365×10^{-10}	3.9440×10^{-10}	7.4003×10^{-10}	1.1720×10^{-6}	1.4728×10^{-6}	1.4524×10^{-6}	
	1.1187×10^{-10}	6.1689×10^{-11}	2.9899×10^{-11}	1.9768×10^{-7}	2.2039×10^{-7}	1.8718×10^{-7}	
Φ_{21}	6.3666×10^{-11}	1.5509×10^{-10}	6.0894×10^{-11}	2.2039×10^{-7}	1.9250×10^{-7}	2.3523×10^{-7}	Φ_{22}
	3.1056×10^{-11}	6.3224×10^{-11}	1.1819×10^{-10}	1.8718×10^{-7}	2.3523×10^{-7}	2.3203×10^{-7}	

Table 7. RMS Error of the Elements of The Unperturbed STM of HEO Orbit

	3.9272×10^{-9}	3.3222×10^{-9}	3.7799×10^{-9}	3.2060×10^{-6}	3.6034×10^{-6}	2.9952×10^{-6}	
Φ_{11}	3.3398×10^{-9}	5.2082×10^{-9}	4.3880×10^{-9}	3.6033×10^{-6}	3.1158×10^{-6}	3.8422×10^{-6}	Φ_{12}
	3.8014×10^{-9}	4.4274×10^{-9}	4.5320×10^{-9}	2.9953×10^{-6}	3.8424×10^{-6}	3.7539×10^{-6}	
	4.4549×10^{-10}	3.4376×10^{-10}	3.0760×10^{-10}	5.1348×10^{-7}	5.7761×10^{-7}	4.7995×10^{-7}	
Φ_{21}	3.5027×10^{-10}	6.6446×10^{-10}	3.9868×10^{-10}	5.7758×10^{-7}	4.9928×10^{-7}	6.1567×10^{-7}	Φ_{22}
	3.1292×10^{-10}	4.0895×10^{-10}	5.3866×10^{-10}	4.7995×10^{-7}	6.1571×10^{-7}	6.0197×10^{-7}	

As shown, there is a loss of computation accuracy in the right half of the STM; however, we believe that improving the series expansion by adaptive time-steps and expansion order will improve the performance across all elements of the STM.

Next, for the J_2 perturbed cases, the results of the 10 orbit periods of the Total Energy Check and the elements of the [$E_{sym.}$] matrix of every step using Analytic Continuation method are compared with the results using ODE45. The results of the four different orbits are plotted against every time step and are shown in Figure.1 to Figure.8.

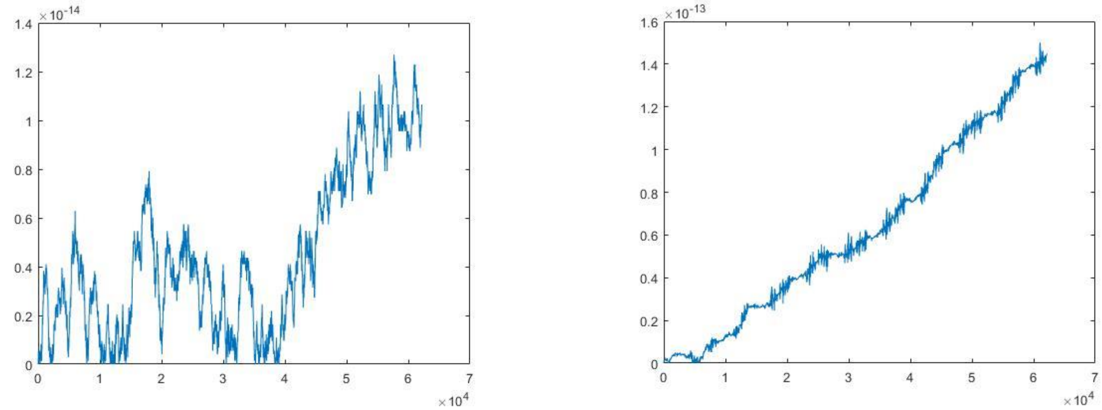


Figure 1(a). Analytic Continuation method **Figure 1(b). ODE45 method**

Figure 1. Total Energy Check vs Time for the LEO Orbit with J_2 Perturbation

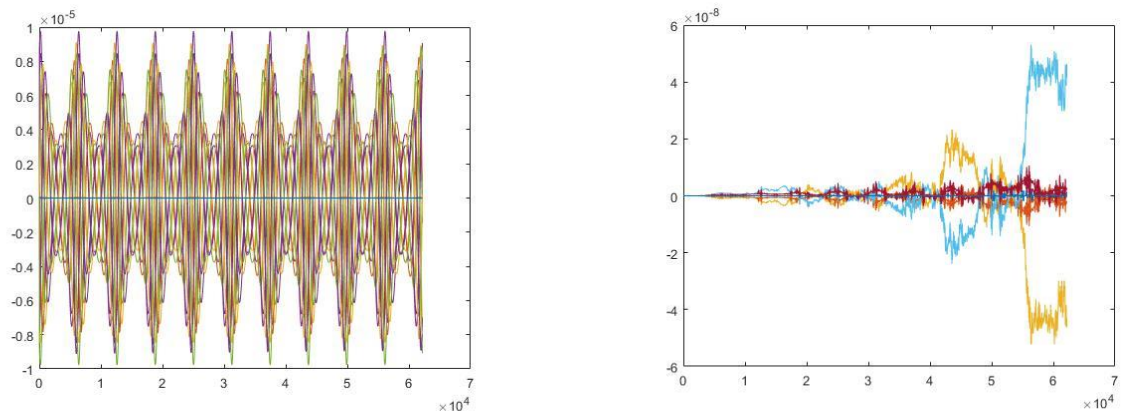


Figure 2(a). Analytic Continuation method **Figure 2(b). ODE45 method**

Figure 2. Symplectic Check vs Time for the STMs of the LEO Orbit with J_2 Perturbation

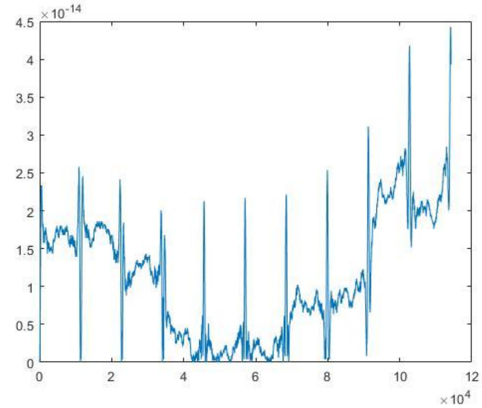


Figure 3(a). Analytic Continuation method

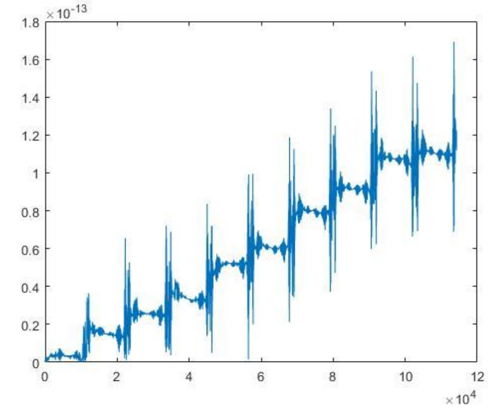


Figure 3(b). ODE45 method

Figure 3. Total Energy Check vs Time for the MEO Orbit with J_2 Perturbation

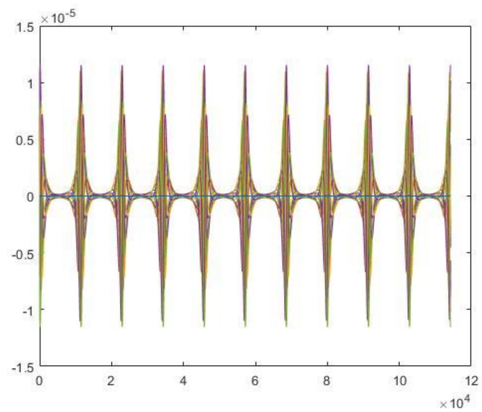


Figure 4(a). Analytic Continuation method

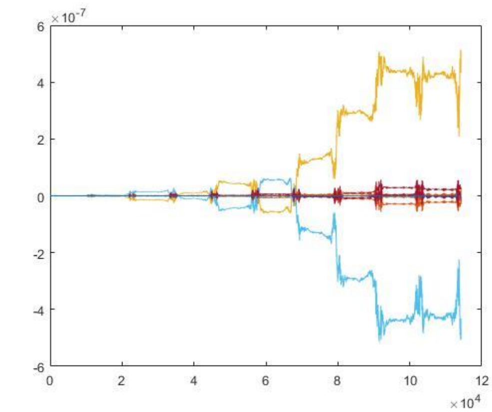
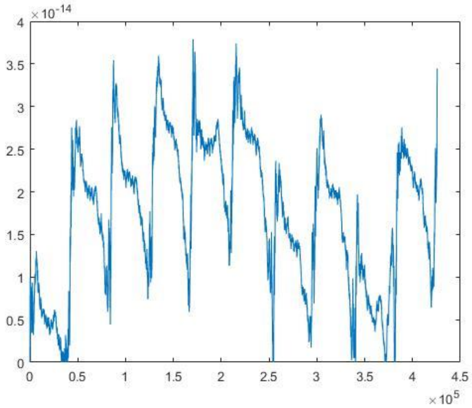


Figure 4(b). ODE45 method

Figure 4. Symplectic Check vs Time for the STMs of the MEO Orbit with J_2 Perturbation



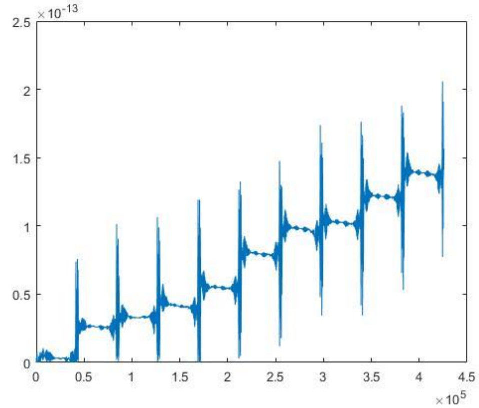
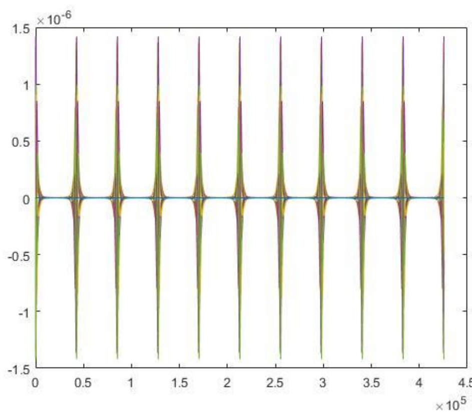


Figure 5(a). Analytic Continuation method

Figure 5(b). ODE45 method

Figure 5. Total Energy Check vs Time for the GTO Orbit with J_2 Perturbation



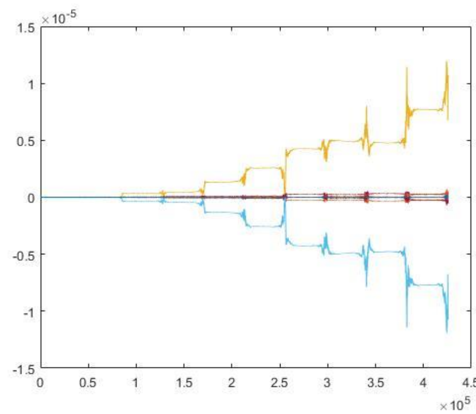


Figure 6(a). Analytic Continuation method

Figure 6(b). ODE45 method

Figure 6. Symplectic Check vs Time for the STMs of the GTO Orbit with J_2 Perturbation

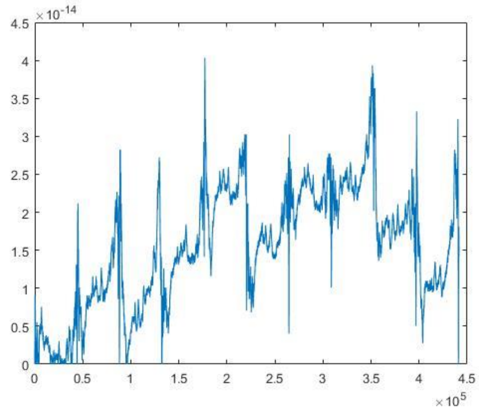


Figure 7(a). Analytic Continuation method

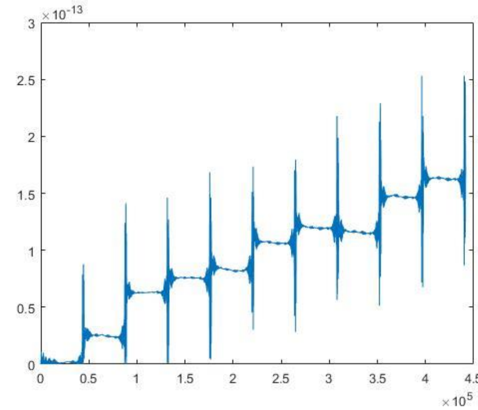


Figure 7(b). ODE45 method

Figure 7. Total Energy Check vs Time for the HEO Orbit with J_2 Perturbation

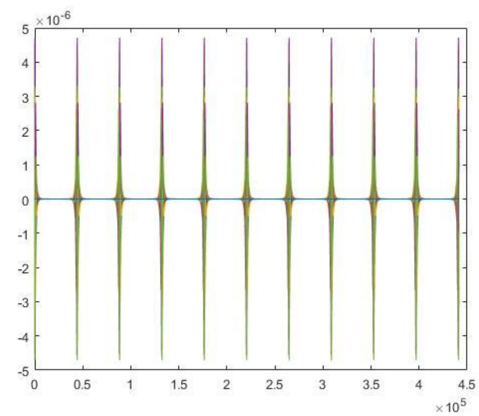


Figure 8(a). Analytic Continuation method

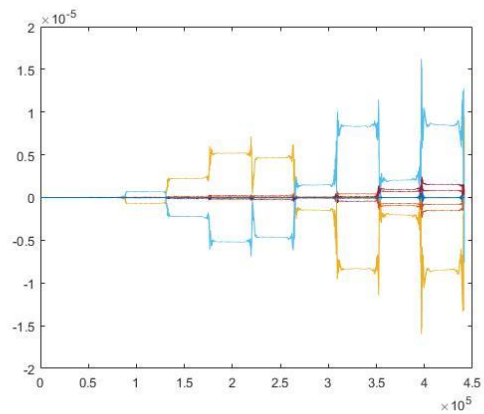


Figure 8(b). ODE45 method

Figure 8. Symplectic Check vs Time for the STMs of the HEO Orbit with J_2 Perturbation

DISCUSSION

As shown from the results, the Analytic Continuation method is providing more accurate results for the perturbed orbit propagation problem across all test orbits. For computing the STMs the new method provides a bounded behavior for the error propagation that is not observed in conventional numerical integration. This phenomenon is currently under further study. Additionally, it is noted that the new approach provided an order of magnitude improvement in accuracy over ODE45 for the GTO and HEO cases; whereas, for the LEO and MEO cases, ODE45 STM is more accurate. However, the accumulation of errors in ODE45 is observed as the number of orbits increases whereas the analytic continuation computed STM error is bounded.

Several additional improvements are currently being explored for the analytic continuation method that includes adaptive time-step and order of expansion. Additionally, the techniques of automatic differentiation are being utilized to improve the efficiency of the present code. These topics will be explored and presented in future works.

CONCLUSION

In this paper, the State Transition Matrix of the perturbed two body problem is derived using higher order Analytic Continuation technique. The derivation is verified by the simulation results of four different types of orbits; LEO, MEO, GTO and HEO, for 10 orbit periods and compared with the simulation results using ODE45. From the comparison it is shown that, the Total Energy Check is at least one digit more accurate than ODE45. In ODE45, the error in the Total Energy Check gradually increases with more revolutions.

The symplectic check of the perturbed STMs using the Analytic Continuation technique shows very interesting characteristics. The plots of the symplectic check show that the errors in Analytic Continuation STM is always bounded in a specific region irrespective of the number of revolutions, whereas the errors in the STMs using ODE45 start increasing with more revolutions. Finally, for the HEO orbit, the STMs using the Analytic Continuation method are at least one digit more accurate than the STMs using ODE45.

Though the present work shows only J_2 perturbation terms, this method can be easily extended to add $J_3 – J_6$ perturbation terms as well as the full spherical harmonics gravity model. Additionally, the method lends itself to readily handle third-body perturbations. Atmospheric drag model can also be included with this work. The aforementioned perturbations are to explore as future extensions of this work. This newly derived method has no singularity and valid for any eccentricity and any type of orbit.

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APPENDIX

The expression for the first six partials of the STM power series are shown here to highlight the recursive relationships shown in Eq. (A1) to Eq. (A10).

$$\frac{\partial \boldsymbol{r}^{(2)}(t)}{\partial \boldsymbol{r}(t)} = \mu\left\{3\frac{g_3(t)}{f(t)}\,\boldsymbol{r}(t)\boldsymbol{r}(t)^{\boldsymbol{T}} - \boldsymbol{I}g_3(t)\right\} \tag{A1}$$

$$\frac{\partial \boldsymbol{r}^{(2)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} = \boldsymbol{0}_{3\times 3} \tag{A2}$$

$$\frac{\partial \boldsymbol{r}^{(3)}(t)}{\partial \boldsymbol{r}(t)} = -\mu\left\{\frac{\partial \boldsymbol{r}(t)}{\partial \boldsymbol{r}(t)} g_3^{(1)}(t) + \boldsymbol{r}(t)\frac{\partial g_3^{(1)}(t)}{\partial \boldsymbol{r}(t)} + \frac{\partial \boldsymbol{r}^{(1)}(t)}{\partial \boldsymbol{r}(t)} g_3(t) + \boldsymbol{r}^{(1)}(t)\frac{\partial g_3(t)}{\partial \boldsymbol{r}(t)}\right\} \tag{A3}$$

$$\frac{\partial \boldsymbol{r}^{(3)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} = -\mu\left\{\boldsymbol{r}(t)\frac{\partial g_3^{(1)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} + \frac{\partial \boldsymbol{r}^{(1)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} g_3(t)\right\} \tag{A4}$$

$$\begin{aligned}\frac{\partial \boldsymbol{r}^{(4)}(t)}{\partial \boldsymbol{r}(t)} = -\mu\Bigg\{&\frac{\partial \boldsymbol{r}(t)}{\partial \boldsymbol{r}(t)} g_3^{(2)}(t) + \boldsymbol{r}(t)\frac{\partial g_3^{(2)}(t)}{\partial \boldsymbol{r}(t)} + 2\left(\frac{\partial \boldsymbol{r}^{(1)}(t)}{\partial \boldsymbol{r}(t)} g_3^{(1)}(t) + \boldsymbol{r}^{(1)}(t)\frac{\partial g_3^{(1)}(t)}{\partial \boldsymbol{r}(t)}\right) \\ &+ \left(\frac{\partial \boldsymbol{r}^{(2)}(t)}{\partial \boldsymbol{r}(t)} g_3(t) + \boldsymbol{r}^{(2)}(t)\frac{\partial g_3(t)}{\partial \boldsymbol{r}(t)}\right)\Bigg\}\end{aligned} \tag{A5}$$

$$\frac{\partial \boldsymbol{r}^{(4)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} = -\mu\left\{\boldsymbol{r}(t)\frac{\partial g_3^{(2)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} + 2\frac{\partial \boldsymbol{r}^{(1)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} g_3^{(1)}(t) + 2\boldsymbol{r}^{(1)}(t)\frac{\partial g_3^{(1)}(t)}{\partial \boldsymbol{r}^{(1)}(t)}\right\} \tag{A6}$$

$$\begin{aligned}\frac{\partial \boldsymbol{r}^{(5)}(t)}{\partial \boldsymbol{r}(t)} = -\mu\Bigg\{&\left(\frac{\partial \boldsymbol{r}(t)}{\partial \boldsymbol{r}(t)} g_3^{(3)}(t) + \boldsymbol{r}(t)\frac{\partial g_3^{(3)}(t)}{\partial \boldsymbol{r}(t)}\right) + 3\left(\frac{\partial \boldsymbol{r}^{(1)}(t)}{\partial \boldsymbol{r}(t)} g_3^{(2)}(t) + \boldsymbol{r}^{(1)}(t)\frac{\partial g_3^{(2)}(t)}{\partial \boldsymbol{r}(t)}\right) \\ &+ 3\left(\frac{\partial \boldsymbol{r}^{(2)}(t)}{\partial \boldsymbol{r}(t)} g_3^{(1)}(t) + \boldsymbol{r}^{(2)}(t)\frac{\partial g_3^{(1)}(t)}{\partial \boldsymbol{r}(t)}\right) + \left(\frac{\partial \boldsymbol{r}^{(3)}(t)}{\partial \boldsymbol{r}(t)} g_3(t) + \boldsymbol{r}^{(3)}(t)\frac{\partial g_3(t)}{\partial \boldsymbol{r}(t)}\right)\Bigg\}\end{aligned} \tag{A7}$$

$$\begin{aligned}\frac{\partial \boldsymbol{r}^{(5)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} = -\mu\Bigg\{&\boldsymbol{r}(t)\frac{\partial g_3^{(3)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} + 3\left(\frac{\partial \boldsymbol{r}^{(1)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} g_3^{(2)}(t) + \boldsymbol{r}^{(1)}(t)\frac{\partial g_3^{(2)}(t)}{\partial \boldsymbol{r}^{(1)}(t)}\right) + 3\boldsymbol{r}^{(2)}(t)\frac{\partial g_3^{(1)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} \\ &+ \frac{\partial \boldsymbol{r}^{(3)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} g_3(t)\Bigg\}\end{aligned} \tag{A8}$$

$$\begin{aligned}\frac{\partial \boldsymbol{r}^{(6)}(t)}{\partial \boldsymbol{r}(t)} = -\mu\Bigg\{&\left(\frac{\partial \boldsymbol{r}(t)}{\partial \boldsymbol{r}(t)} g_3^{(4)}(t) + \boldsymbol{r}(t)\frac{\partial g_3^{(4)}(t)}{\partial \boldsymbol{r}(t)}\right) + 4\left(\frac{\partial \boldsymbol{r}^{(1)}(t)}{\partial \boldsymbol{r}(t)} g_3^{(3)}(t) + \boldsymbol{r}^{(1)}(t)\frac{\partial g_3^{(3)}(t)}{\partial \boldsymbol{r}(t)}\right) \\ &+ 6\left(\frac{\partial \boldsymbol{r}^{(2)}(t)}{\partial \boldsymbol{r}(t)} g_3^{(2)}(t) + \boldsymbol{r}^{(2)}(t)\frac{\partial g_3^{(2)}(t)}{\partial \boldsymbol{r}(t)}\right) + 4\left(\frac{\partial \boldsymbol{r}^{(3)}(t)}{\partial \boldsymbol{r}(t)} g_3^{(1)}(t) + \boldsymbol{r}^{(3)}(t)\frac{\partial g_3^{(1)}(t)}{\partial \boldsymbol{r}(t)}\right) \\ &+ \left(\frac{\partial \boldsymbol{r}^{(4)}(t)}{\partial \boldsymbol{r}(t)} g_3(t) + \boldsymbol{r}^{(4)}(t)\frac{\partial g_3(t)}{\partial \boldsymbol{r}(t)}\right)\Bigg\}\end{aligned} \tag{A9}$$

$$\begin{aligned}\frac{\partial \boldsymbol{r}^{(6)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} = -\mu\Bigg\{&\boldsymbol{r}(t)\frac{\partial g_3^{(4)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} + 4\left(\frac{\partial \boldsymbol{r}^{(1)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} g_3^{(3)}(t) + \boldsymbol{r}^{(1)}(t)\frac{\partial g_3^{(3)}(t)}{\partial \boldsymbol{r}^{(1)}(t)}\right) + 6\boldsymbol{r}^{(2)}(t)\frac{\partial g_3^{(2)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} \\ &+ 4\left(\frac{\partial \boldsymbol{r}^{(3)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} g_3^{(1)}(t) + \boldsymbol{r}^{(3)}(t)\frac{\partial g_3^{(1)}(t)}{\partial \boldsymbol{r}^{(1)}(t)}\right) + \frac{\partial \boldsymbol{r}^{(4)}(t)}{\partial \boldsymbol{r}^{(1)}(t)} g_3(t)\Bigg\}\end{aligned} \tag{A10}$$