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An Extension of the Minimum Principle with Application to Switched Linear Systems

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Abstract— **In this paper we address the problem of optimal switching for switched linear systems. The uniqueness of our approach lies in casting the switching action as multiple control inputs. This allows us to apply the concept of *Pontryagin's Minimum Principle* in solving the optimal control problem. There is no restriction on the switching sequence or the number of switchings although the later is bound by the sampling interval. This is in contrast to search based algorithms where a fixed number of switchings is set apriori. We solve the optimization problem using an iterative scheme where a two point boundary value problem is solved. Simulation results are provided to support the proposed method.**

I. INTRODUCTION

A switched system is one where the system can switch between multiple modes or state variable descriptions during its operation. With the objective of enhancing the performance of the system, a switching law is invoked that determines the switching times and mode transitions. Switched systems have found many applications, such as air traffic management, manufacturing systems, chemical processes, embedded automotive controllers, and vibration control.

The problems related to optimal control and optimal switching are fundamentally challenging and have attracted the attention of many researchers in recent years. Egerstedt et al. [4] developed an algorithm for computing the minimum number of switchings for the transition of a linear system from one state to another. To consider variable number of switchings, Egerstedt et al. [5] developed an approach based on the gradient of the cost functional. In [9], Xu and Antsakalis proposed the idea of two stage optimization and in [10] they used nonlinear optimization techniques to determine the switching instants. Although these results provide open-loop solutions, there exist a few results that provide switching laws based on state variable measurements, [2], [6]. Most of the results in the literature, [1], [5], [7], [10], for example, are based on preselecting a finite number of switchings and optimization of the switching times against a cost function. For the approaches adopted in these papers, the result of optimization can be improved if the number of switchings are allowed to increase. However, since these methods rely on a search algorithm to determine the switching times, the cost of computation can increase significantly with increase in the number of switchings.

Recently Bengea [3] et al. proposed a solution to the optimal switching problem where the number of switchings is unconstrained and the switching sequence is not specified apriori. The authors solve the optimization problem for a larger family of systems of which the switched system is a subset. In this paper we address the optimal switching problem through an extension of the *Pontryagin's Minimum Principle* since the minimum principle cannot be applied directly to our problem. This is because the system switches between discrete behaviors and all convex combinations of these behaviors are not realized. Our method results in an offline solution and requires solving a two-point boundary value problem by applying an iterative scheme. The optimal control is formulated for fixed final time with no restrictions on the number of switchings, switching intervals, and switching sequence.

II. PROBLEM STATEMENT

Consider the linear system which can switch between the following N time-invariant modes:

$$\begin{array}{ll} S_1: & \dot{X} = A_1 X, \\ S_2: & \dot{X} = A_2 X, \\ \vdots & \quad \vdots \\ S_N: & \dot{X} = A_N X \end{array} \tag{1}$$

where $X \in R^n$ and $X(t_0) = X_0$. We wish to optimally switch between these N modes to minimize the cost functional

$$J = \frac{1}{2}\int_{t_0}^{t_f} X^T Q X dt \tag{2}$$

where t_0 is the fixed initial time, t_f is the fixed final time, and Q is a symmetric positive definite matrix. It is assumed that the switching sequence is not specified apriori and there is no restriction imposed on the number of switchings. The total number of switchings, however, is assumed to be finite since it will be constrained by the frequency of switching which can atmost be equal to the frequency of sampling for any digital implementation.

By treating switching as a control action, the switched system can be described by the form

$$\begin{aligned} \dot{X} = f(X,U) = A_1 X u_1 + A_2 X\,(1-u_1)\,u_2 + \cdots \\ + A_{N-1} X \prod_{i=1}^{i=N-2}(1-u_i)\,u_{N-1} + A_N X \prod_{i=1}^{i=N-1}(1-u_i) \end{aligned} \tag{3}$$

where the effect of switching is captured by the control input vector $U = [u_1 \ u_2 \ \ \cdots u_{N-1}]^T \in R^{N-1}$, with u_i, $i = 1, 2, \cdots, (N-1)$, assuming integer values of zero or one, *i.e.* $u_i \in \{0, 1\}$. Indeed,

$$\begin{aligned} \dot{X} &= A_i X, \quad \textit{iff} \;\; u_i = 1, \; u_j = 0, \\ &\quad \forall \, i, j = 1, 2, \cdots, (N-1), \; \forall \, j < i \\ \dot{X} &= A_N X, \; \textit{iff} \;\; u_i = 0, \;\; \forall \, i = 1, 2, \cdots, (N-1) \end{aligned} \tag{4}$$

Although $u_i \in \{0, 1\}$, $i = 1, 2, \cdots, (N-1)$ can result in $2^{(N-1)}$ combinations of the inputs, these combinations essentially result in the N distinct system descriptions of Eq.4. The optimal control problem can now be posed as follows: For the dynamical system described by Eq.3, determine time trajectories of u_i, $u_i \in \{0, 1\}$, $i = 1, 2, \cdots, (N-1)$, that minimizes the cost functional in Eq.2.

III. Extension of the Minimum Principle

A. *Optimal Switching Law*

For the optimal control problem posed in Section II, the control inputs were constrained to assume discrete integer values and therefore *Pontryagin's Minimum Principle* cannot be directly applied. We solve the problem with the help of the following theorem.

Theorem 1: The linear system which can switch between the N time-invariant modes in Eq.(1) can be described by Eq.(3) with admissible inputs $u_i \in \{0, 1\}$, $i = 1, 2, \cdots, (N-1)$. The switching condition that minimizes the cost functional in Eq.(2) is

$$\min_{k \in \{1, 2, \cdots, N\}} [\lambda(t)^T A_k \, X(t)], \quad \forall \;\; t \in [t_0, t_f] \tag{5}$$

where $\lambda(t) \in R^N$ are the co-states and are the solutions of

$$\begin{aligned} \dot{\lambda} &= -QX - \Big[A_1^T \lambda u_1 + A_2^T \lambda \, (1-u_1) \, u_2 + \cdots \\ &+ A_{N-2}^T \lambda \prod_{i=1}^{i=N-2} (1-u_i) \, u_{N-1} + A_N^T \lambda \prod_{i=1}^{i=N-1} (1-u_i) \Big] \end{aligned} \tag{6}$$

with the boundary condition $\lambda(t_f) = 0$.

Proof: Consider the dynamical system in Eq.(3), but relax the constraints on the the control inputs by allowing them to vary continously between 0 and 1, *i.e.*, $u_i \in [0, 1]$. This modification allows us to apply *Pontryagin's Minimum Principle*, as discussed below.

By augmenting the cost functional J in Eq.(2) with the constraint in Eq.(3), we have

$$J = \int_{t_0}^{t_f} \left[\frac{1}{2} X^T Q X + \lambda^T \left(f(X, U) - \dot{X} \right) \right] dt \tag{7}$$

We define the *Hamiltonian*, H, as follows

$$H(X, \lambda, U) = \frac{1}{2} X^T Q X + \lambda^T f(X, U) \tag{8}$$

The state equations are obtained as

$$\dot{X} = (\partial H / \partial \lambda) = f(X, U) \tag{9}$$

and the co-state equations can be shown to be those in Eq.(6) with boundary condition $\lambda(t_f) = 0$. To determine the optimal control inputs, $u_i = u_i^*$, we apply the *Minimum Principle* to get

$$H(X, \lambda, U^*) \leq H(X, \lambda, U) \quad \forall \;\; t_0 \leq t \leq t_f \tag{10}$$

where $U^* = [u_1^* \ u_2^* \ \cdots \ u_{N-1}^*]^T$ represents the optimal control input vector. To apply the condition in Eq.(10), we minimize H with respect to u_i, $i = 1, 2, \cdots, (N-1)$. By inspecting Eqs.(3) and (8), we deduce that we effectively need to minimize

$$\begin{aligned} \overline{H} \, (X, \lambda, U) &= \lambda^T \Big[A_1 X u_1 + A_2 X \, (1-u_1) \, u_2 + \cdots \\ &+ A_{N-1} X \prod_{i=1}^{i=N-2} (1-u_i) \, u_{N-1} + A_N X \prod_{i=1}^{i=N-1} (1-u_i) \Big] \end{aligned} \tag{11}$$

with respect to u_i, $u_i \in [0, 1]$, $i = 1, 2, \cdots, (N-1)$. If we assume the optimal inputs, u_i^*, $i = 2, 3, \cdots, (N-1)$, to be known, we are left with minimizing Eq.(11) with respect to u_1 only, i.e.,

$$\begin{aligned} \min_{u_1 \in [0,1]} \lambda^T \Big[& A_1 X u_1 + A_2 X \, (1-u_1) \, u_2^* + \cdots \\ & + A_{N-1} X (1-u_1) \prod_{i=2}^{i=N-2} (1-u_i^*) \, u_{N-1}^* \\ & + A_N X (1-u_1) \prod_{i=2}^{i=N-1} (1-u_i^*) \Big] \end{aligned} \tag{12}$$

Since the expression in Eq.(12) is linear in u_1, u_1^* will be equal to 0 or 1. Similarly, if the optimal inputs u_i^*, $i = 1, 3, 4, \cdots, (N-1)$ are known, we can minimize Eq.(11) with respect to u_2,

$$\begin{aligned} \min_{u_2 \in [0,1]} \; \lambda^T \Big[& A_1 X u_1^* + A_2 X \, (1-u_1^*) \, u_2 + \cdots \\ & + A_{N-1} X (1-u_1^*)(1-u_2) \prod_{i=3}^{i=N-2} (1-u_i^*) \, u_{N-1}^* \\ & + A_N X (1-u_1^*)(1-u_2) \prod_{i=3}^{i=N-1} (1-u_i^*) \Big] \end{aligned} \tag{13}$$

to determine u_2^*. Again, the expression in Eq.(13) is linear in u_2 and hence u_2^* will be equal to 0 or 1. Proceeding in the same manner for the other control inputs, we can conclude that although we relaxed the constraint on the control inputs and allowed them to belong to the set $[0, 1]$, the optimal inputs take values 0 or 1 only, *i.e.*, $u_i^* \in \{0, 1\}$, $i = 1, 2, \cdots, (N-1)$. Since $\{0, 1\} \subset [0, 1]$, we claim $u_i^* \in \{0, 1\}$, $i = 1, 2, \cdots, (N-1)$ for the original dynamical system in Eq.3 with admissible inputs $u_i \in \{0, 1\}$. From Eq.(4) and the subsequent discussion, we also know that the $2^{(N-1)}$ combinations of u_i^*, $u_i^* \in \{0, 1\}$, $i = 1, 2, \cdots, (N-1)$, result in the N distinct system modes of Eq.(1) and these modes are each associated with the following N control input vectors,

$$\begin{aligned} U_1 &= [1 \ 0 \ \cdots \ 0 \ 0]^T, \qquad U_2 = [0 \ 1 \ \cdots \ 0 \ 0]^T, \cdots \\ U_{N-1} &= [0 \ 0 \ \cdots \ 0 \ 1]^T, \quad \; U_N = [0 \ 0 \ \cdots \ 0 \ 0]^T \end{aligned} \tag{14}$$

Therefore,

$$\overline{H} \, (X, \lambda, U^*) = \min \left\{ \overline{H}_1, \overline{H}_2, \cdots, \overline{H}_N \right\} \quad \forall \;\; t \in [t_0, t_f] \tag{15}$$

where

$$\overline{H}_k = \overline{H}\left(X, \lambda, U_k\right), \qquad \forall\ k = 1, 2, \cdots, N \tag{16}$$

Equations (10), (11), (14), (15), and (16) together imply that the optimal switching condition is

$$\min_{k \in \{1,2,\cdots,N\}} [\lambda(t)^T A_k\, X(t)], \qquad \forall\ t \in [t_0, t_f] \tag{17}$$

and this concludes the proof. §§§

To determine the optimal solution from the switching condition in Eq.(17), we need to solve a two point boundary value problem. This is because N initial conditions are only known for the states X. The remaining N boundary conditions are obtained from $\lambda(t_f) = 0$, which results from free final states $X(t_f)$. The two point boundary value problem can be solved using an iterative scheme such as the *Relaxation Method* [8], which we have adopted here.

IV. Simulation

Consider the two-mode switched linear system with

$$A_1 = \begin{bmatrix} -1 & 1 \\ -18 & -5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \tag{18}$$

Let $t_0 = 0$ sec, $t_f = 2.5$ sec, $X(0) = [-0.2\ \ 0.6]^T$, and the weighting matrix Q to be the identity matrix. The state and co-state equations can be written in terms of a single control input, u_1, as follows

$$\dot{X} = A_1 X u_1 + A_2 X (1 - u_1), \tag{19}$$

$$\dot{\lambda} = -QX - A_1^T \lambda u_1 - A_2^T \lambda (1 - u_1) \tag{20}$$

The two-point boundary conditions are

$$X(0) = [-0.2\ \ 0.6]^T,\ \lambda(2.5) = [0\ \ 0]^T \tag{21}$$

The optimal switching condition is

$$\min_{k \in \{1,2\}}\ [\lambda(t)^T A_k\, X(t)], \quad \forall\ t\ \in [0, 2.5] \tag{22}$$

The above two point boundary value problem was solved

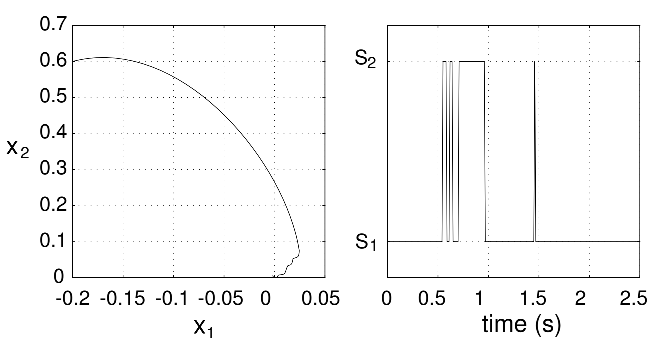

Fig. 1. State trajectory and optimal switching sequence

using the *Relaxation Method* [8]. In the relaxation method, ordinary differential equations (ODEs) are approximated by finite difference equations (FDEs) on a mesh of points that span the domain of interest and the correct solution is obtained iteratively, starting with a guess solution. For our problem, the FDEs were generated through first order Taylor series approximation. We used 251 mesh points with a mesh size of 0.01 sec. The initial guess solution was chosen as $x_1(t) = -0.2$, $x_2(t) = 0.6$, $\lambda_1(t) = -2e^{-5t}$ and $\lambda_2(t) = 2e^{-5t}$. The state trajectories and the switching sequence and instants are plotted in Fig.1. S_1 and S_2 are representative of the two system modes A_1 and A_2 respectively. Both A_1 and A_2 have all stable eigenvalues. However, switching is performed to optimize the cost functional which is computed to be $J_{opt} = 0.0488$. This is lower than the cost of remaining in S_1, which is computed to be $J_1 = 0.0489$, and the cost of remaining in S_2, which is computed to be $J_2 = 0.8977$.

V. Conclusion

In this paper we have developed an approach for solving the optimal switching problem for switched linear systems, where the switching action is modeled as multiple control inputs and an extension of the *Pontryagin's Minimum Principle* is used in deriving the optimal switching laws. The switching sequence and the number of switchings are not selected apriori. The proposed method leads to offline optimization and relies on the iterative *Relaxation Method* for solving a two point boundary value problem. Simulation results are provided to support the optimization method developed above.

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