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Exponential Stabilization of the Rolling Sphere: Stability Analysis

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Abstract— **In an earlier paper [3], we addressed the problem of stabilization of the rolling sphere about any desired configuration. For the controller proposed in [3], we establish in this paper global stability of the desired configuration and exponential convergence of trajectories to this configuration from a large and well defined set in the configuration space. For configurations that lie outside the set, we define simple maneuvers to move them within the set in finite time such that the sphere can be exponentially converged to the desired configuration thereafter. Our theoretical claims are validated using simulations.**

I. INTRODUCTION

The problem feedback stabilization pose unique challenges for nonholonomic systems since standard nonlinear control methods do not lend themselves well toward stabilization to an equilibrium state. This problem has been overcome by development of strategies that may be classified under smooth time-varying stabilization [6], piecewise-smooth time-invariant stabilization [1], [8], and hybrid stabilization [7]. A special class of nonholonomic systems is the class of two-input nilpotentizable systems that can be transformed into a "chained form". The necessary and sufficient conditions for existence of a feedback transformation to chained-form was provided by Murray [4]. The chained-form, by its very structure and construction, allows the development of control algorithms. It has been shown that the kinematic model of the sphere cannot be converted to chained-form and hence stabilization strategies need to be custom designed.

The nonholonomic control problem of the rolling sphere has been addressed only by a few. Date, et al., [2] used the time-state control form to design a controller that was shown to converge all states to the equilibrium state but stability of the equilibrium was not adequately investigated. Oriolo and Vendittelli [5] recently showed that the equilibrium can be stabilized through iterative application of an open-loop control law designed for nilpotent approximation of the system. They proposed a two part algorithm which involves converging three states of the sphere first and then executing repetitive closed trajectories of these states to steer the two other states closer to their desired coordinates. The main drawback of the algorithm is that in the presence of perturbations, the first part of the algorithm must be redone before executing the second part.

A stabilizing algorithm for the rolling sphere was proposed by Mukherjee and Das [3]. The control strategy utilizes a sequence of alternate linear and circular motions of the sphere to achieve complete reconfiguration. In Section II of this paper we revisit the main results established in [3]. In Section III we design preliminary maneuvers to enable convergence of the sphere to the desired configuration from certain special initial configurations. The complete reconfiguration strategy is constructed in Section IV. Stability and exponential convergence is established in Section V. Simulation results are presented in Section VI and concluding remarks are provided in Section VII.

II. BACKGROUND

We describe the sphere using the two Cartesian coordinates of its center and three coordinates that describe its orientation. We denote the Cartesian coordinates of the sphere center by $Q \equiv (x, y)$ and adopt the z-y-z Euler angle sequence (α, θ, ϕ) to represent the orientation of the sphere, as shown in Fig.1. Complete reconfiguration is achieved by bringing P to the vertically upright position, and R, which then lies on the diametrical circle in the xy plane, to lie on the positive x axis which is shown in Fig.1(b). Denoting the angular

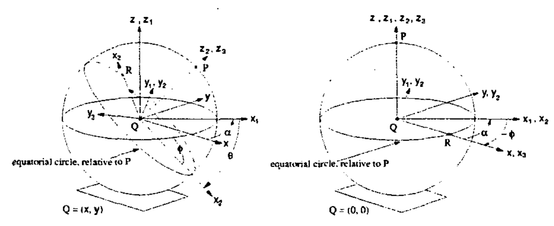

Fig. 1. (a) An arbitrary configuration of the sphere, (b) desired configuration of the sphere.

velocities of the sphere about the x_1, y_1, z_1 axes as ω_x^1, ω_y^1, ω_z^1, respectively, the state equations for $\omega_z^1 = 0$ are written as follows

$$\begin{aligned} \dot{x} &= \omega_y^1 \cos\alpha + \omega_x^1 \sin\alpha \\ \dot{y} &= \omega_y^1 \sin\alpha - \omega_x^1 \cos\alpha \end{aligned} \qquad \begin{aligned} \dot{\theta} &= \omega_y^1 \\ \dot{\alpha} &= -\omega_x^1 \cot\theta \\ \dot{\beta} &= \omega_x^1 \tan\frac{\theta}{2} \end{aligned} \qquad (1)$$

where $\beta = \alpha + \phi$. Complete reconfiguration is achieved by satisfying

$$x = 0, \quad y = 0, \quad \theta = 0, \quad \beta = 0 \qquad (2)$$

We define the following control actions

- (A) $\omega_y^1 \neq 0, \quad \omega_x^1 = 0$
- (B) $\omega_x^1 \neq 0, \quad \omega_y^1 = 0, \qquad 0 < \theta < \pi/2$

It is shown in Fig.2 that control actions (A) and (B) cause the sphere to move along straight line and circular arc segments respectively. It can be shown that complete reconfiguration implies

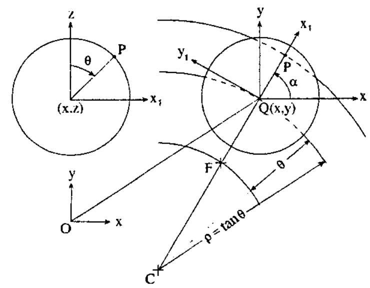

Fig. 2. Motion of the sphere under control actions (A) and (B)

$$(CF, CO, \beta) \equiv (0,0,0) \iff (x,y,\theta,\beta) \equiv (0,0,0,0) \tag{3}$$

Thus the specific choice of Euler angles has allowed us to achieve reconfiguration by converging fewer variables, C, F, and β, to the origin. We now define a fundamental theorem on which the reconfiguration algorithm is based

Theorem 1: (Dual-Point Theorem) Let C and F be two points in the xy plane with origin at O. Suppose $\psi = \angle OCF$ is acute, and let $(CF/CO) = n$. If ψ satisfies the condition

$$0 \leq \psi < \cos^{-1}(1/n) \quad \text{for } n \in (1,\infty) \tag{4a}$$

$$0 \leq \psi < \cos^{-1}(n) \quad \text{for } n \in (0,1) \tag{4b}$$

then there exists a point C' on the extended line CF such that for $\psi' = \angle OC'F$, $0 \leq \psi' \leq \pi$,

$$\begin{aligned} &(C'F/C'O) = n && \cdots\cdots (i) \\ &0 < (C'O/CO) < 1 && \cdots\cdots (ii) \\ &\psi' > \psi && \cdots\cdots (iii) \end{aligned} \tag{5}$$

We define the following three basic maneuvers:

Definition 1: (DPT maneuver) In reference to Theorem 1, we define a "Dual-Point Tuck" (DPT) maneuver as control action (A) that moves the sphere such that C moves to C'.

Definition 2: (RS maneuver) Following a DPT maneuver, a control action (B) that moves the sphere to restore ψ' to ψ is defined as a "Restoring-Sweep" (RS) maneuver.

It can be shown that the RS maneuver can be executed in multiple options. A specific choice of RS maneuver, called the CRS maneuver, is defined below

Definition 3: (CRS maneuver) Among four choices for an RS maneuver, the Compensating and Restoring Sweep (CRS) minimizes the absolute value of β.

We recall the "Third Reconfiguration Theorem" in [3]:

Theorem 2: (Third Reconfiguration Theorem) Let $(x_0, y_0, \theta_0, \alpha_0, \beta_0)$ be the initial configuration of the sphere that satisfies $n \in (0,1) \cup (1,\infty)$ and $0 < \theta_0 \leq (\pi/2 - \epsilon)$. If β_0 lies in the range

$$(3\pi - \zeta)(1 - \sec\theta_0) \leq \beta_0 \leq -(3\pi - \zeta)(1 - \sec\theta_0) \tag{6}$$

the sphere can be completely reconfigured by a PPS maneuver followed by repeated application of CRS-DPT pairs with $\psi \in [\Psi, \zeta]$. If β_0 lies outside the range, the sphere can be reconfigured completely by applying a TO maneuver followed by a PPS maneuver and followed by repeated application of CRS-DPT pairs with $\psi \in [\max(\bar{\psi}, \Psi), \zeta]$. The definitions and discussion of ζ, Ψ, $\bar{\psi}$, and PPS maneuver, can be referred in [3].

Definition 4: (TO Maneuver) At the initial time, if $\theta_0 < \theta^*$, where

$$|\beta_0| = -(3\pi - \pi/2)(1 - \sec\theta^*) \tag{7}$$

a control action (A) that increases the value of θ to θ^* is defined as a "Tuck-Out" (TO) maneuver.

III. SPECIAL CASES

In this section we investigate the cases where $n = 0$, $n = 1$, $n = \infty$, the case where n is undefined, and $\theta_0 > (\pi/2 - \epsilon)$. For each of these special cases, we provide a strategy comprised of at-most two maneuvers that change the configuration of the sphere to one that satisfies $n \in (0,1) \cup (1,\infty)$ and $0 < \theta \leq (\pi/2 - \epsilon)$, such that Theorem 2 can be subsequently applied for complete reconfiguration.

$\underline{n = 1}$: This special case occurs when $CF = CO \neq 0$. We investigate this case separately for two cases, namely, (a) $\theta_0 < \theta^*$, and (b) $\theta_0 \geq \theta^*$. For case (a) we propose a TO maneuver. For case (b), we change n by:

1. first using control action (B) to make points O, C, and F collinear, and in that order.
2. then using control action (A) to change n.

Since $\theta_0 \geq \theta^*$, Eq.(6) holds good prior to step (1). The complete range of β_0 in Eq.(6) equals

$$(6\pi - 2\zeta)(\sec\theta_0 - 1) \geq 5\pi(\sec\theta_0 - 1) \tag{8}$$

whereas the maximum change in β_0 due to control action (B) in step (1) is $|\Delta\beta|_{max} = 2\pi(\sec\theta_0 - 1)$. Since the range of β_0 prior to step (1) is more than twice the absolute change of β_0 that can occur in step (1), we will select cw or ccw sweep, as shown in Fig.3(a), such that Eq.(6) is satisfied also after the execution of step (1). In step (2), we decrease θ to the value of θ^* if $\theta > \theta^*$. If $\theta = \theta^*$, we change the value of θ to θ_1, where θ_1 is given by the relation

$$\theta_1 = k_1 \theta^*, \quad 1 < k_1 \leq k_{max} \tag{9}$$

where, $k_{max} = (\pi/2 - \epsilon)/\theta^*_{max}$. In the above equation, ϵ is arbitrarily small and k_1 is chosen such that the sphere does not end in a configuration with $n = \infty$ when O, C, and F are collinear.

$\underline{n=\infty}$: This case occurs when $CO = 0$, and $CF \neq 0$. As in the case of $n = 1$, we investigate this case separately for the cases: $\theta_0 < \theta^*$, $\theta_0 = \theta^*$, and $\theta_0 > \theta^*$. If $\theta_0 < \theta^*$, we propose to apply the TO maneuver. This changes n while θ_0 increases to θ^*. If $\theta_0 > \theta^*$, we reduce θ_0 to θ^* which again changes n. If $\theta_0 = \theta^*$, we increase θ to θ_1, where

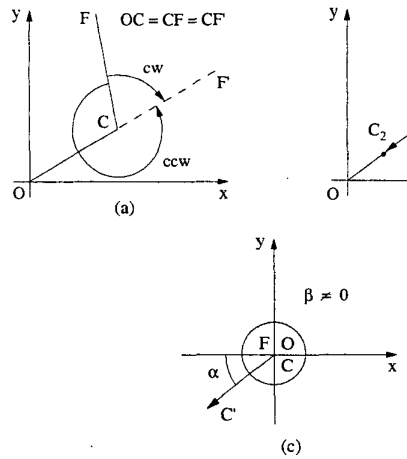

Fig. 3. Initial maneuvers for special cases (a) $n = 1$, (b) $n = 0$, (c) n is undefined

$$\theta_1 = k_\infty \theta^* \quad k_\infty = (\pi/2 - \epsilon)/\theta^*_{max} \tag{10}$$

$\underline{n=0}$: In this case, $CF = 0$, and $CO \neq 0$, as shown in Fig.3(b). This implies that $\theta_0 = 0$. If $\beta_0 \neq 0$, we use a TO maneuver to increase θ_0 to θ^*. We choose to move C along line OF but we have two choices, namely, move C away from the origin to C_1, or move C towards the origin to C_2. We choose to move C away from the origin to eliminate the possibility of ending in the special cases, $n = 1$ or $n = \infty$.

If $\beta_0 = 0$, we use control action (A) to change the value of n and to increase the value of θ from zero to θ_1, where,

$$\theta_1 = \begin{cases} k_0\sqrt{x_0^2 + y_0^2} & \text{if } k_0\sqrt{x_0^2 + y_0^2} \leq (\pi/2 - \epsilon) \\ (\pi/2 - \epsilon) & \text{if } k_0\sqrt{x_0^2 + y_0^2} > (\pi/2 - \epsilon) \end{cases} \tag{11}$$

where x_0 and y_0 are the initial Cartesian coordinates of the sphere center Q. By choosing θ_1 in accordance with Eq.(11) we ensure that the sphere does not end in a configuration with $\theta > (\pi/2 - \epsilon)$. Furthermore, similar to the case where $\beta_0 \neq 0$, control action (A) is chosen to move C away from the origin along OF.

$\underline{n \text{ is undefined}}$: This case occurs when $CF = CO = 0$, and $\beta_0 \neq 0$. For this case we apply a TO maneuver to increase θ from zero to θ^*. Since α is initially undefined, we use the TO maneuver to move C in an arbitrary direction, as shown in Fig.3(c). At the end of the TO maneuver, the sphere will be at a configuration where $n = 1$. Then we follow the steps outlined above for the case $n = 1$ to change the value of n.

$\underline{\theta_0 > (\pi/2 - \epsilon)}$: In this case we use control action (A) to reduce θ to $\theta_1 \in [\theta^*_{max}, (\pi/2 - \epsilon)]$ such that

$$\theta_1 = k_\theta \theta_0, \quad (\theta^*_{max}/\theta_0) \leq k_\theta < 1 \tag{12}$$

By properly choosing the value of k_θ, we will avoid the sphere from ending in a configuration where n has a value of 0, 1, or ∞, or is undefined.

IV. COMPLETE RECONFIGURATION ALGORITHM

We now assimilate the reconfiguration strategy in Theorem 2 with the strategies proposed for coping with the special cases to arrive at a complete reconfiguration algorithm for all initial conditions. For convergence to the desired configuration $(x, y, \theta, \beta) = (0, 0, 0, 0)$, we ascertain that there does not exist any infinite loop generated by the special cases with the help of Fig.4 where all configurations of the sphere are categorized under eight configuration sets, S_i, $i = 1, 2, \cdots 8$

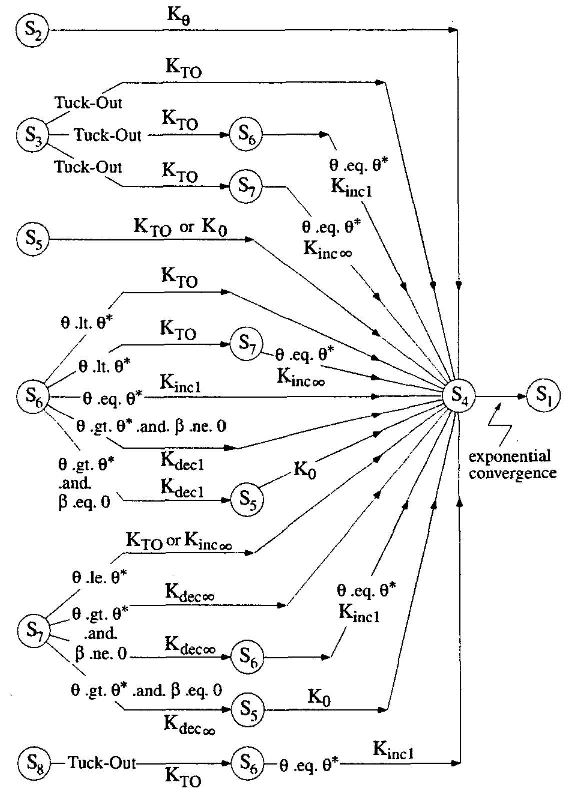

Fig. 4. Transition between all configuration sets of the sphere

$S_1 : \{X \,|\, x = y = \theta = \beta = 0\}$
$S_2 : \{X \,|\, \theta > (\pi/2 - \epsilon)\}$
$S_3 : \{X \,|\, \theta < \theta^* < (\pi/2 - \epsilon), n \in (0, 1) \cup (1, \infty)\}$
$S_4 : \{X \,|\, \theta^* \leq \theta \leq (\pi/2 - \epsilon), n \in (0, 1) \cup (1, \infty)\}$
$S_5 : \{X \,|\, n = 0\}$
$S_6 : \{X \,|\, n = 1\}$
$S_7 : \{X \,|\, n = \infty\}$
$S_8 : \{X \,|\, n \text{ is undefined}\}$

and all possible transitions between the configuration sets are considered based on our discussion in [3] and Section III. It is clear from Fig.4 that all configurations of the sphere in the set $S_2 \cup S_3 \cup S_5 \cup S_6 \cup S_7 \cup S_8$ reach S_4 in at-most two steps and all configurations in S_4 reach the desired configuration S_1. Thus our convergence algorithm converges the sphere to the desired configuration from all initial configurations.

V. STABILITY ANALYSIS

A. Coordinate Transformation

In this section we prove that the equilibrium configuration $(x, y, \theta, \beta) = (0, 0, 0, 0)$ is also stable. We use the transformation

$$R = x^2 + y^2, \qquad \sigma = \tan^{-1}(y/x), \qquad \Theta = \theta^2 \tag{13}$$

to obtain the new kinematic model

$$\begin{aligned} \dot{R} &= 2\sqrt{R}\left\{\sin(\alpha-\sigma)\,\omega_x^1 + \cos(\alpha-\sigma)\,\omega_y^1\right\} \\ \dot{\sigma} &= \frac{1}{\sqrt{R}}\left\{-\cos(\alpha-\sigma)\,\omega_x^1 + \sin(\alpha-\sigma)\,\omega_y^1\right\} \\ \dot{\Theta} &= 2\sqrt{\Theta}\,\omega_y^1 \\ \dot{\alpha} &= -\omega_x^1 \cot\sqrt{\Theta} \\ \dot{\beta} &= \omega_x^1 \tan(\sqrt{\Theta}/2) \end{aligned} \tag{14}$$

In this kinematic model, the desired configuration of the sphere is defined by $(R, \Theta, \beta) \equiv (0, 0, 0)$. We will prove stability of the equilibrium configuration by defining $\|X\| = (R^2 + \Theta^2 + \beta^2)^{1/2}$ and showing that for each $\kappa > 0$ there exists a $\delta = \delta(\kappa)$ such that

$$\|X(0)\| \le \delta \implies \|X(t)\| \le \kappa \tag{15}$$

We will also prove exponential stability for all $X \subset S_4$.

B. Exponential Stability of the Equilibrium Configuration

We note that CF and CO remain constant during RS and PPS maneuvers. For DPT maneuvers, their magnitudes may increase during a portion of the maneuver if $n \in (0, 1)$, but decrease overall. CF and CO, at the end of a DPT maneuver, are always a fraction of their values at the start of the maneuver. If we assume that each RS-DPT pair requires equal time to execute, the variation in CF and CO can be approximated by curves that decrease in geometric progression over equal intervals of time. Such curves have exponential rates of decay and we can write

$$\begin{aligned} CF(t) &= CF(0)\exp[-\gamma_1 t] \\ CO(t) &= CO(0)\exp[-\gamma_1 t] \end{aligned} \quad \gamma_1 = \frac{1}{\Delta t}\ln\left(\frac{1}{r}\right) \tag{16}$$

where Δt is the time interval required for execution of each RS-DPT pair. Since successive RS-DPT pairs will require lesser time to execute, a conservative choice for Δt is the time required for the first RS-DPT pair.

Since $CF = (\tan\theta - \theta)$ decreases exponentially in time, so do θ, $\tan\theta$, and $(\sec\theta - 1)$. Specifically, using the Comparison Lemma we obtain

$$\begin{aligned} \theta(t) &\le \theta(0)\exp[-\gamma_1\gamma_2 t] \\ f[\theta(t)] &\le f[\theta(0)]\exp[-(\gamma_1/3)t], \; f(\theta) = \tan\theta \\ g[\theta(t)] &\le g[\theta(0)]\exp[-(2\gamma_1/3)t], \; g(\theta) = \sec\theta - 1 \end{aligned} \tag{17}$$

$$\gamma_2(\epsilon) = \left[(\cot\theta/\theta) - \cot^2\theta\right]_{(\pi/2-\epsilon)} > 0 \tag{18}$$

Theorem 3: (Exponential Stability) The reconfiguration algorithm in Theorem 2 renders the equilibrium configuration $(R, \Theta, \beta) \equiv (0, 0, 0)$ exponentially stable for $X \subset S_4$.

Proof: Consider an arbitrary configuration of the sphere in Fig.2 for $X \subset S_4$. Using the triangle inequality, the definition of R in Eq.(13), the relation $CQ = \tan\theta$, and Eqs.(16) and (17) we can write

$$R(t) \le 2\left[CO^2(0) + CQ^2(0)\right]\exp[-(2\gamma_1/3)t] \tag{19}$$

Using the triangle identity we again write

$$CO^2(0) \le 2\left[R(0) + CQ^2(0)\right] \tag{20}$$

By substituting Eq.(20) in Eq.(19), we get

$$R^2(t) \le 8\left[4R^2(0) + 9CQ^4(0)\right]\exp[-(4\gamma_1/3)t] \tag{21}$$

Since $0 < \theta \le (\pi/2 - \epsilon)$, $CQ \le c_1\sqrt{\Theta}$, where $c_1 = [\tan(\pi/2-\epsilon)/(\pi/2-\epsilon)]$. Thus Eq.(21) yields

$$R^2(t) \le 8\left[4R^2(0) + 9c_1^4\Theta^2(0)\right]\exp[-(4\gamma_1/3)t] \tag{22}$$

From Theorem 2 we know

$$|\beta(t)| \le (3\pi - \zeta)\left[\sec\theta(t) - 1\right] \le 3\pi\left[\sec\theta(t) - 1\right]$$

Using Eq.(17) we can therefore write

$$\beta^2(t) \le 9\pi^2\left[\sec\theta(0) - 1\right]^2\exp[-(4\gamma_1/3)t] \tag{23}$$

Again, since $0 < \theta \le (\pi/2 - \epsilon)$, $[\sec\theta(t) - 1] \le c_2\Theta$, where $c_2 = [\sec(\pi/2-\epsilon) - 1]/(\pi/2-\epsilon)^2$. Thus Eq.(23) yields

$$\beta^2(t) \le 9\pi^2 c_2^2\Theta^2(0)\exp[-(4\gamma_1/3)t] \tag{24}$$

By combining Eqs.(22), (24), and the expression for $\theta(t)$ in Eq.(17), we get using Holder inequality

$$\begin{aligned} \|X(t)\|^2 &\le \left[c_3R^2(0) + c_4\Theta^2(0)\right]\exp[-\gamma_3 t] \\ &\le c_5\|X(0)\|^2\exp[-\gamma_3 t] \end{aligned} \tag{25}$$

where, $c_3 = 32$, $c_4 = 72c_1^4 + 9\pi^2c_2^2 + 1$, $c_5 = \max[c_3, c_4]$, and $\gamma_3 = 2\gamma_1.\min[(2/3), \gamma_2]$. From Eq.(25) we claim exponential stability of the equilibrium configuration. ◇◇◇

Theorem 4: (Stability) The reconfiguration algorithm in Section IV guarantees uniform stability of the equilibrium configuration $(R, \Theta, \beta) \equiv (0, 0, 0)$.

Proof: We only provide an outline of the proof for brevity. At the initial time, let $X \subset S_3$. This implies that the sphere will undergo a TO maneuver, as shown in Fig.5. For this maneuver using triangle inequality

$$\sqrt{R(t)} \le \sqrt{R(0)} + (\theta^* - \theta_0) \le \sqrt{R(0)} + \theta^* \tag{26}$$

Also, from Eq.(7) we know

$$|\beta_0| = \frac{5\pi}{2}(\sec\theta^* - 1) \Rightarrow \theta^2 \le \theta^{*2} \le \left(\frac{4}{5\pi}\right)|\beta_0| \tag{27}$$

Since a TO maneuver does not change the value of $\beta = \beta_0$, we can show using Eqs.(26) and (27)

$$\|X(t)\| \le K_{TO}\|X(0)\|, \qquad K_{TO} = 2\sqrt{2} \tag{28}$$

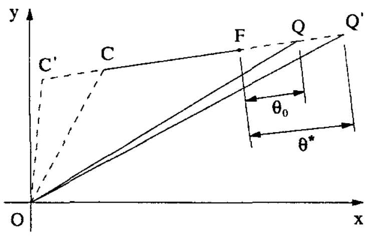

Fig. 5. Motion of the sphere during a TO maneuver

The constant K_{TO} is shown in Fig.4 for transitions from S_3 to S_4, S_6, and S_7. The constants associated with transition from the other configuration sets can be similarly derived based on our discussion in Section III. All of them are greater than unity and are shown in Fig.4 and derived in the Appendix. For transition from $S_2 \cup S_3 \cup S_5 \cup S_6 \cup S_7 \cup S_8$ to S_4 we can therefore claim

$$\|X(t)\| \leq K_{max} \|X(0)\| \tag{29}$$

$$K_{max} = \max\{K_\theta, K_{TO}K_1, K_O K_2, K_{inc1}K_{dec\infty}\},$$

$$K_1 = \max\{K_{inc1}, K_{inc\infty}\}, K_2 = \max\{K_{dec1}, K_{dec\infty}\}$$

For the transition from S_4 to S_1, we have from Eq.(25)

$$\|X(t)\| \leq \sqrt{c_5}\, \|X(0)\| \tag{30}$$

By combining Eqs.(29) and (30), we get

$$\|X(t)\| \leq K_{max}\sqrt{c_5}\, \|X(0)\| \tag{31}$$

Thus for any $\kappa > 0$, we can choose $\delta = (1/K_{max}\sqrt{c_5})\kappa$ such that Eq.(15) will be satisfied. $\diamond\diamond\diamond$

VI. SIMULATION RESULTS

We present two simulations for which the initial conditions of the sphere are as follows

$$(x\ y\ \theta\ \alpha\ \beta) \equiv (5.50\ \ 1.50\ \ 1.30\ \ \pi/2\ \ 2.50) \tag{32a}$$

$$(x\ y\ \theta\ \alpha\ \beta) \equiv (0.00\ \ 0.00\ \ 0.00\ \ \pi/4\ \ 0.60) \tag{32b}$$

where the units are in meters and radians.

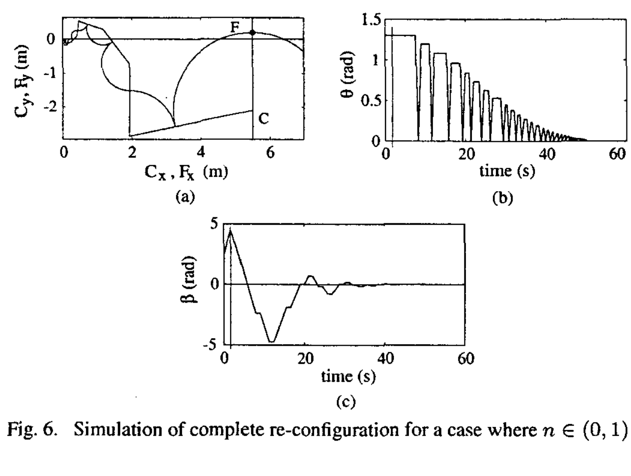

Fig. 6. Simulation of complete re-configuration for a case where $n \in (0,1)$

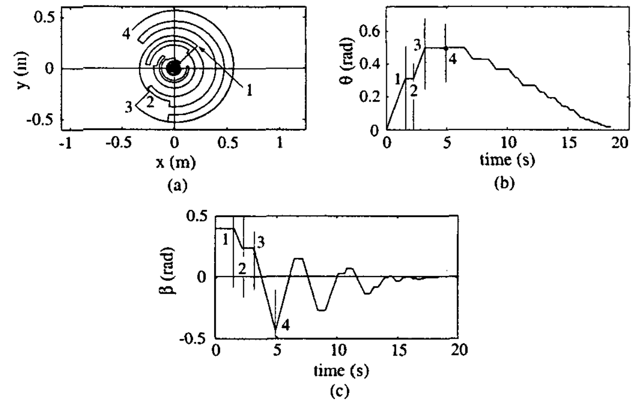

Fig. 7. Simulation of complete reconfiguration representative of special cases n = undefined and $n = 1$, and the general case $n \in (1, \infty)$

The simulation in Fig.6 corresponds to the case $n = 0.39 \in (0,1)$ where the initial value of β satisfies Eq.(6). To apply Theorem 2, we choose $\psi = 0.58 \in [\max(\bar{\psi}, \Psi), \cos^{-1}(n)]$. The trajectories of C and F are shown in Fig.6(a). In this figure, the circular motions of point F corresponds to the PPS and CRS maneuvers and the linear motions of point C corresponds to DPT maneuvers. The trajectory of θ and β are shown in Fig.6(b) and (c) respectively. The vertical lines indicate the end of the PPS maneuver. The CRS and DPT maneuvers can also be inferred from these figures since θ remains constant during CRS maneuvers and β remains constant during DPT maneuvers. Fig.6 confirms that x, y, θ and β all converge to zero and hence the sphere is completely reconfigured.

The simulation in Fig.7 is representative of the special cases n = undefined and $n = 1$, and the general case $n \in (1, \infty)$. Since n = undefined at the initial time, the sphere first performs a TO maneuver - this results in the configuration marked 1 in Figs.7(a), (b), and (c). The sphere then acquires a configuration where $n = 1$ and $\theta = \theta^* = 0.31$. The sphere now sweeps to align O, C and F and reaches the configuration 2. Subsequently it increases θ to θ_1 defined in Eq.(9) for the choice of $k_1 = 1.6$. This changes θ to 0.5, n to 1.84, and the sphere configuration to the point marked 3. A PPS maneuver is now performed and the sphere moves to the configuration 4. With choice of $\psi = 0.65 \in [\max(\bar{\psi}, \Psi), \cos^{-1}(1/n)]$, complete reconfiguration is finally achieved with a sequence of CRS-DPT pairs.

The fundamental difference in the trajectories of θ in Figs.6(b) and 7(b) is attributed to the discontinuous change in α during DPT maneuvers for $n \in (0,1)$ as opposed to a continuous change in α for $n \in (1, \infty)$.

VII. CONCLUSION

In this paper we addressed the problem of feedback stabilization of a sphere rolling without slipping on a flat surface. We have outlined the fundamental convergence algorithm for the set of initial conditions satisfying $n \in (0,1) \cup (1,\infty)$, which is detailed in [3]. We have identified the specific set of initial configurations which lie outside this space and designed few preliminary maneuvers to converge them within the general category within finite time. We have shown the global stability of the desired configuration and have also shown exponential convergence from a large and well-defined set in configuration space. The efficacy of our algorithm is adequately demonstrated by numerical simulations.

VIII. ACKNOWLEDGEMENT

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X. APPENDIX

In this section prove that the preliminary maneuvers described in Section III guarantee stability of the equilibrium configuration. Stability analysis for TO maneuvers can be found under Theorem 4 and hence not repeated here.

$\underline{n=1}$: When $\theta_0 \geq \theta^*$, as discussed in Section III, we initially apply a control action (B) for which Eq.(25) is valid and hence we can write Eq.(30). Next we apply a control action (A). If $\theta_0 > \theta^*$ we decrease θ from θ_0 to θ^*. Applying triangle inequality we have

$$\sqrt{R(t)} \leq \sqrt{R(0)} + \theta_0 - \theta^* \Rightarrow R(t) \leq 2(R(0) + \Theta(0)) \qquad (A.1)$$

Also since $\theta \leq \theta_0$ and $\beta = \beta_0$, we can show that

$$\|X(t)\| \leq 3\|X(0)\| \qquad (A.2)$$

If $\theta_0 = \theta^*$ we increase θ from θ_0 to $k_1\theta_0$, where k_1 is defined in Eq.(9). Applying triangle inequality we have

$$\sqrt{R(t)} \leq \sqrt{R(0)} + k_1\theta_0 - \theta_0 \qquad (A.3)$$

Also since $\theta \leq k_1\theta_0$ and $\beta = \beta_0$ we can show that

$$\|X(t)\| \leq \sqrt{8(k_1-1)^4 + k_1^4 + 9}\, \|X(0)\| \qquad (A.4)$$

Thus, combining Eqs.(30), (A.2), (A.4), we have

$$\begin{array}{ll} \theta > \theta^* : & \|X(t)\| \leq K_{dec1}\|X(0)\| \\ \theta = \theta^* : & \|X(t)\| \leq K_{inc1}\|X(0)\| \end{array} \qquad (A.5)$$

$$K_{dec1} = 3\sqrt{c_5}, \ K_{inc1} = \sqrt{8(k_1-1)^4 + k_1^4 + 9\sqrt{c_5}}$$

$\underline{n=\infty}$: The stability analysis for the preliminary maneuvers for this special case is similar to the $n = 1$ case discussed above. We state the end results here:

$$\begin{array}{ll} \theta > \theta^* : & \|X(t)\| \leq K_{dec\infty}\|X(0)\| \\ \theta = \theta^* : & \|X(t)\| \leq K_{inc\infty}\|X(0)\| \end{array} \qquad (A.6)$$

$$K_{dec\infty} = 3, \ K_{inc\infty} = \sqrt{8(k_\infty-1)^4 + k_\infty^4 + 9}$$

$\underline{n=0}$: This special case requires a control action (A), as explained in Section III, when $\beta_0 = 0$. For this preliminary maneuver, $\Theta \leq {k_0}^2 R(0)$, $\beta = \beta_0 = 0$. Also, from triangle inequality we have

$$R(t) \leq (k_0+1)^2 R(0) \qquad (A.7)$$

Hence we have

$$\|X(t)\| \leq K_0\|X(0)\|, \ K_0 = \sqrt{(k_0+1)^4 + {k_0}^4} \qquad (A.8)$$

$\underline{n = \text{undefined}}$: The preliminary control action for this special case is a TO maneuver which leads to $n = 1$. The stability analysis for $n = 1$ is discussed above and hence not repeated here.

$\underline{\theta_0 > (\frac{\pi}{2} - \epsilon)}$: The preliminary maneuver is a control action (A) to decrease θ from θ_0 to $\theta_f \leq (\pi/2 - \epsilon)$. From triangle inequality we have

$$R(t) \leq 2(R(0) + \Theta(0)) \qquad (A.9)$$

Also, $\theta \leq \theta_0$ and $\beta = \beta_0$, hence we have

$$\|X(t)\| \leq K_\theta\|X(0)\|, \ K_\theta = 3 \qquad (A.10)$$