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Technical communique

Optimally switched linear systems☆

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Abstract

In this paper we address the problem of optimal switching for switched linear systems. The uniqueness of our approach lies in describing the switching action by multiple control inputs. This allows us to embed the switched system in a larger family of systems and apply *Pontryagin's Minimum Principle* for solving the optimal control problem. This approach imposes no restriction on the switching sequence or the number of switchings. This is in contrast to search based algorithms where a fixed number of switchings is set a priori. In our approach, the optimal solution can be determined by solving the ensuing two-point boundary value problem. Results of numerical simulations are provided to support the proposed method.

Keywords: Switched system; Pontryagin's minimum principle; Optimal switching

1. Introduction

A switched system is described by multiple modes or state variable descriptions and a switching condition that triggers transitions between the modes. Typically, the switching condition is based on state and/or time events and aims to enhance the performance of the system. Switched systems have found many applications such as in air traffic management, manufacturing systems, chemical processes, embedded automotive controllers, and vibration control.

For switched systems with linear component systems, the problems of optimal control and optimal switching have attracted the attention of several researchers in recent years. For example, Egerstedt, Wardi, and Delmotte (2003), developed an algorithm for computing the minimum number of switchings for transition from one state to another and Xu and Antsakalis (2004), addressed the problem of determining the switching instants. These results, like most of the results in the literature, provide open-loop solutions and are based on pre-selecting a finite number of switchings and optimization of the switching instants against a cost functional. For these open-loop solutions, the results of optimization can be improved if the number of switchings is allowed to increase but since these methods use a search algorithm, the cost of computation increases dramatically.

Some of the recent papers on the optimal switching problem provide state feedback solutions. Assuming a known and finite switching sequence, a state feedback controller was designed by Giua, Seatzu, and Van Der Me (2001); the controller relies on a numerically constructed switching table that indicates points in the state space where switchings should occur. The optimization problem was later generalized by Bemporad, Giua, and Seatzu (2002a) by taking both the switching instants and the switching sequence as decision variables. Using discrete- and continuous-time techniques, a master–slave algorithm was developed but this algorithm does not guarantee convergence to the global optimum and is only valid for small perturbations around the initial state. A global optimal solution that can determine the optimal feedback law for all initial states was proposed by the same group (Bemporad, Giua, & Seatzu, 2002b), but the computation cost of this algorithm is of the

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order Ns^2, where N is the number of switchings and s is the number of system modes. The algorithm also requires one of the system modes to be strictly Hurwitz.

In this communique, we generalize the optimization problem further by treating the switching sequence, number of switchings, and switching instants, all as decision variables. In deviation from the traditional approach, we solve the optimization problem as an optimal control problem by first describing the switching action using multiple control inputs, then embedding the switched system into a larger family of system for application of Pontryagin's Minimum Principle, and finally applying the results for the larger family of systems to the embedded switched system. A two-point boundary value problem is solved to obtain an open-loop solution where the decision to switch and the new system mode are both decided based on a simple algebraic condition. It should be noted that the idea of embedding a switched system into a larger family of systems has been proposed (Bengea & DeCarlo, 2005) and the application of the Minimum Principle to hybrid systems has appeared in the literature (Branicky, Borkar, & Mitter, 1998; Cassandras, Pepyne, & Wardi, 2001; Sussmann, 1999). Also, in the absence of additional conditions, global optimality cannot be guaranteed since the Minimum Principle provides only necessary conditions.

2. Problem statement

Consider the linear system which can switch between the following N time-invariant modes:

$$\begin{aligned} S_1 &: \dot{X} = A_1 X, \\ S_2 &: \dot{X} = A_2 X, \\ &\vdots \\ S_N &: \dot{X} = A_N X \end{aligned} \tag{1}$$

where $X \in R^n$ and $X(t_0) = X_0$. We wish to switch between these N modes to minimize the cost functional

$$J = \frac{1}{2}\int_{t_0}^{t_f} X^{\mathrm{T}} Q X \,\mathrm{d}t \tag{2}$$

where t_0 is the fixed initial time, t_f is the fixed final time, and Q is a symmetric positive definite matrix. It is assumed that the switching sequence is not specified a priori and there is no restriction imposed on the number of switchings.

The switched system can be described by the form

$$\begin{aligned} \dot{X} = f(X, U) = {} & A_1 X u_1 + A_2 X (1-u_1) u_2 + \cdots \\ & + A_{N-1} X \prod_{i=1}^{i=N-2} (1-u_i) u_{N-1} + A_N X \prod_{i=1}^{i=N-1} (1-u_i) \end{aligned} \tag{3}$$

where the effect of switching is captured by the control input vector $U = \begin{bmatrix} u_1 & u_2 & \cdots & u_{N-1} \end{bmatrix}^{\mathrm{T}} \in R^{N-1}$, with u_i, $i = 1, 2, \ldots, (N-1)$, assuming integer values of zero or one, *i.e.* $u_i \in \{0, 1\}$. Indeed,

$$\begin{aligned} \dot{X} &= A_i X, \quad \text{iff } u_i = 1, u_j = 0, \forall j < i \\ &\qquad \forall i = 1, 2, \ldots, (N-1) \\ \dot{X} &= A_N X, \quad \text{iff } u_i = 0, \forall i = 1, 2, \ldots, (N-1). \end{aligned} \tag{4}$$

Although $u_i \in \{0, 1\}$, $i = 1, 2, \ldots, (N-1)$ can result in $2^{(N-1)}$ combinations of the inputs, these combinations essentially result in the N distinct system descriptions of Eq. (4). This is illustrated with the example of the four-mode system

$$\begin{aligned} \dot{X} = f(X, U) = {} & A_1 X u_1 + A_2 X (1-u_1) u_2 \\ & + A_3 X (1-u_1)(1-u_2) u_3 \\ & + A_4 X (1-u_1)(1-u_2)(1-u_3) \end{aligned}$$

where $U = \begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}^{\mathrm{T}} \in R^3$. It can be verified that there are four distinct modes, namely,

$$\begin{aligned} \dot{X} &= A_1 X, \quad \text{when } U = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}, U = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^{\mathrm{T}}, \\ &\qquad U = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^{\mathrm{T}}, U = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}} \\ \dot{X} &= A_2 X, \quad \text{when } U = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}, U = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^{\mathrm{T}}, \\ \dot{X} &= A_3 X, \quad \text{when } U = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}, \\ \dot{X} &= A_4 X, \quad \text{when } U = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}. \end{aligned}$$

Furthermore, the four distinct modes are associated with the three unit control input vectors and the null vector $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^{\mathrm{T}}$, $\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$, and $\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$, respectively.

3. Application of the Minimum Principle

The optimal control problem can be posed as follows: For the dynamical system described by Eq. (3), determine time trajectories of u_i, $u_i \in \{0, 1\}$, $i = 1, 2, \ldots, (N-1)$, that minimizes the cost functional in Eq. (2). Since the control inputs are constrained to assume discrete integer values, we cannot apply *Pontryagin's Minimum Principle* directly. To alleviate this problem, we embed our switched system into a larger family of systems by allowing the inputs to vary continuously in the range [0, 1]. By showing that optimal inputs for the larger family of systems belong to the set {0, 1}, we claim that application of *Pontryagin's Minimum Principle* to the larger family of systems leads to optimal inputs for the switched system. This is discussed in the proof of the following theorem.

Theorem. *The linear system which can switch between the N time-invariant modes in Eq. (1) can be described by Eq. (3) with admissible inputs $u_i \in \{0, 1\}$, $i = 1, 2, \ldots, (N-1)$. The switching condition that minimizes the cost functional in Eq. (2) is*

$$k(t) = \underset{k \in \{1,2,\ldots,N\}}{\arg\min} \, [\lambda(t)^{\mathrm{T}} A_k X(t)], \quad \forall t \in [t_0, t_f] \tag{5}$$

where $\lambda(t) \in R^N$, the co-states, are the solutions of

$$\dot{\lambda} = -QX - \Big[A_1^{\mathrm{T}} \lambda u_1 + A_2^{\mathrm{T}} \lambda (1-u_1) u_2 + \cdots$$

$$+ A_{N-1}^{\mathrm{T}} \lambda \prod_{i=1}^{i=N-2} (1-u_i)\, u_{N-1} + A_N^{\mathrm{T}} \lambda \prod_{i=1}^{i=N-1} (1-u_i) \Bigg] \tag{6}$$

with the boundary condition $\lambda(t_f) = 0$.

Proof. Consider the dynamical system in Eq. (3), but allow the control inputs to vary continuously between 0 and 1, *i.e.*, $u_i \in [0, 1]$. This modification allows us to apply *Pontryagin's Minimum Principle*, as discussed below.

By augmenting the cost functional J in Eq. (2) with the constraint in Eq. (3), we have

$$J = \int_{t_0}^{t_f} \left[\frac{1}{2} X^{\mathrm{T}} Q X + \lambda^{\mathrm{T}} \left(f(X, U) - \dot{X} \right) \right] \mathrm{d}t.$$

We define the *Hamiltonian*, H, as follows

$$H(X, \lambda, U) = \frac{1}{2} X^{\mathrm{T}} Q X + \lambda^{\mathrm{T}} f(X, U). \tag{7}$$

The state equations are obtained as

$$\dot{X} = (\partial H / \partial \lambda) = f(X, U)$$

and the co-state equations can be shown to be those in Eq. (6) with boundary condition $\lambda(t_f) = 0$. To determine the optimal control inputs, $u_i = u_i^*$, we apply the *Minimum Principle* to get

$$H(X, \lambda, U^*) \le H(X, \lambda, U) \quad \forall t_0 \le t \le t_f \tag{8}$$

where $U^* = \left[u_1^* \quad u_2^* \quad \cdots \quad u_{N-1}^* \right]^{\mathrm{T}}$ represents the optimal control input vector. To apply Eq. (8), we minimize H with respect to u_i, $i = 1, 2, \ldots, (N-1)$. By inspecting Eqs. (3) and (7), we deduce that we effectively need to minimize

$$\overline{H}(X, \lambda, U) = \lambda^{\mathrm{T}} \Bigg[A_1 X u_1 + A_2 X (1-u_1) u_2 + \cdots + A_{N-1} X \prod_{i=1}^{i=N-2} (1-u_i)\, u_{N-1} + A_N X \prod_{i=1}^{i=N-1} (1-u_i) \Bigg] \tag{9}$$

with respect to u_i, $u_i \in [0, 1]$, $i = 1, 2, \ldots, (N-1)$. Eq. (9) can be expressed as

$$\overline{H}(X, \lambda, U) = a_1 v_1 + a_2 v_2 + \cdots + a_N v_N$$

where $a_i = \lambda^{\mathrm{T}} A_i X$, for $i = 1, 2, \ldots, N$, and

$$\begin{aligned} v_1 &= u_1 \\ v_2 &= u_2 (1-u_1) \\ &\vdots \\ v_{N-1} &= u_{N-1} \prod_{i=1}^{i=N-2} (1-u_i) \\ v_N &= \prod_{i=1}^{i=N-1} (1-u_i). \end{aligned} \tag{10}$$

From the above equation it can be shown that $v_i \in [0, 1]$, since $u_i \in [0, 1]$, and $\sum_{i=1}^{N} v_i = 1$. Thus,

$$\min \overline{H}(X, \lambda, U) = a_k = \min \{a_1, a_2, \ldots, a_N\} \tag{11}$$

which is achieved by choosing $v_k = 1$ and $v_i = 0$, $\forall i \ne k$. It can be shown using Eq. (10) that this can be achieved by

$$u_k^* = 1, \qquad u_i^* = 0, \qquad u_j^* \in [0, 1], \qquad \forall i \le k, \forall j \ge k$$

which includes the input combination $u_k^* = 1$, $u_i^* = 0$, $\forall i \ne k$. Although we relaxed the constraint on the control inputs and allowed them to belong to the set [0, 1], optimality is achieved with all inputs taking on values of zero or unity, *i.e.*, $u_i^* \in \{0, 1\}$, $i = 1, 2, \ldots, (N-1)$. Since $\{0, 1\} \subset [0, 1]$, we claim $u_i^* \in \{0, 1\}$, $i = 1, 2, \ldots, (N-1)$ for the original dynamical system in Eq. (3) with admissible inputs $u_i \in \{0, 1\}$. Furthermore, Eq. (11) implies Eq. (5) and this concludes the proof. □

4. Numerical solution

To determine the optimal solution, we need to solve a two-point boundary value problem since N initial conditions are known for the states X whereas the remaining N boundary conditions are $\lambda(t_f) = 0$, a consequence of free final states $X(t_f)$. We solve the two-point boundary value problem using the *Relaxation Method* (Press, Teukolsky, Vetterling, & Flanney, 1992), where ordinary differential equations are approximated by finite difference equations (FDEs) on a mesh of points that span the domain of interest and the optimal[1] solution is obtained iteratively, starting with a guess solution. At each iteration of the relaxation method, the algorithm computes an error norm, *err*, by summing the absolute value of all corrections weighted by scale factors which are measures of the typical size of each variable. When this norm is less than a specified parameter *conv*, the solution is assumed to have converged to the true solution. At each iteration, a factor *slowc* is used to apply only a part of the corrections generated by the algorithm. When *err* > *slowc* only a fraction of the corrections is applied and when *err* < *slowc* the entire correction is applied.

4.1. Example 1

Consider the two-mode switched linear system with

$$A_1 = \begin{bmatrix} -10 & 5 \\ 3 & -2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 3 & 1 \\ 7 & -20 \end{bmatrix}.$$

Let $t_0 = 0$ s, $t_f = 2.5$ s, $X(0) = \begin{bmatrix} 1 & 6 \end{bmatrix}^{\mathrm{T}}$, and the weighting matrix Q to be the identity matrix. The state and co-state equations can be written in terms of a single control input, u_1, as follows

$$\begin{aligned} \dot{X} &= A_1 X u_1 + A_2 X (1-u_1), \\ \dot{\lambda} &= -QX - A_1^{\mathrm{T}} \lambda u_1 - A_2^{\mathrm{T}} \lambda (1-u_1). \end{aligned}$$

The two-point boundary conditions are

$$X(0) = \begin{bmatrix} 1 & 6 \end{bmatrix}^{\mathrm{T}}, \qquad \lambda(2.5) = \begin{bmatrix} 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

[1] Although our optimality condition indicates that infinite switchings can occur, numerical simulations require a finite time interval between switchings and hence provide sub-optimal trajectories.

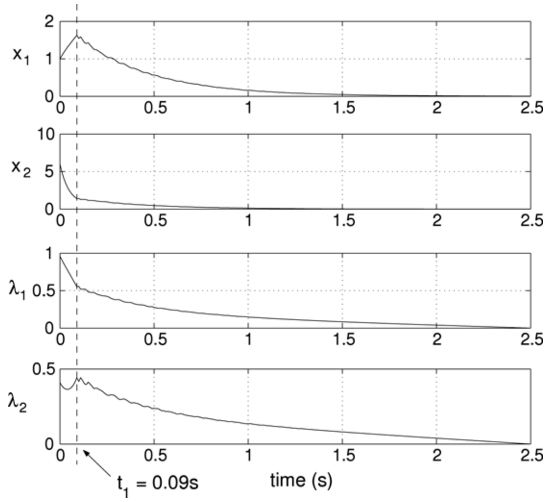

Fig. 1. State and co-state trajectories: Two-mode system.

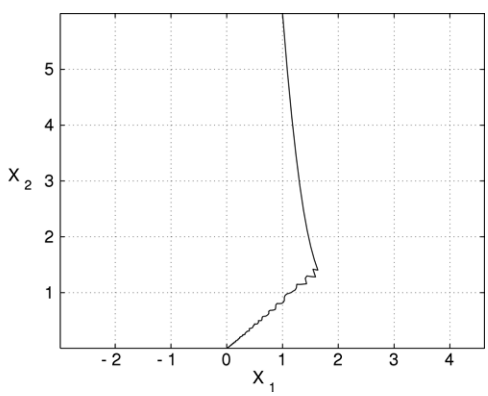

Fig. 2. Phase portrait: Two-mode system.

and the optimal switching condition is

$$\min_{k\in\{1,2\}} [\lambda(t)^{\mathrm{T}} A_k\, X(t)], \quad \forall t \in [0, 2.5].$$

For the *Relaxation Method* (Press et al., 1992), we used 251 mesh points with a mesh size of 0.01 s. A total of 9248 iterations were required for *err* to reduce from an initial value of 189.823 to *conv* which was chosen to be 0.003.

The state and co-state trajectories are plotted in Fig. 1. The phase portrait is shown in Fig. 2 and the optimal switching times are shown in Fig. 3. It is evident from Fig. 3 that the system dwells in mode S_2 till $t_1 = 0.09$ s and switches modes regularly thereafter. The frequent switching and the initial dwelling in mode S_2 can be explained by the eigenvalues of A_1 (mode S_1) and A_2 (mode S_2), which are

$$A_1 = \begin{cases}\mu_{11} = -11.57\\ \mu_{12} = -0.43\end{cases}, \qquad A_2 = \begin{cases}\mu_{21} = 3\\ \mu_{22} = -20.3.\end{cases}$$

In spite of stable eigenvalues of A_1, the initial dwelling in mode S_2 and frequent switching to S_2 from S_1 beyond t_1 can be explained by the large negative value of μ_{22}. The frequent transition to S_1 beyond $t = t_1$ can be attributed to the positive eigenvalue of S_2, namely, $\mu_{21} = 3$. The value of J under optimal switching was computed to be 1.0138.

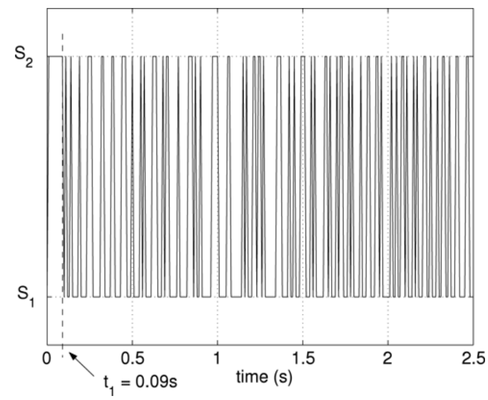

Fig. 3. Optimal switching: Two-mode system.

4.2. Example 2

Consider the three-mode switched system

where, $A_1 = \begin{bmatrix}-4 & 5\\ 3 & -9\end{bmatrix}, \qquad A_2 = \begin{bmatrix}7 & 1\\ 7 & -15\end{bmatrix},$

$A_3 = \begin{bmatrix}-15 & 2\\ 4 & 6\end{bmatrix}.$

Let $t_0 = 0$ s, $t_f = 2.5$ s, $X(0) = \begin{bmatrix}1 & 6\end{bmatrix}^{\mathrm{T}}$, and the weighting matrix Q to be the identity matrix. The state and co-state equations can be written in terms of the switching input $U = \begin{bmatrix}u_1 & u_2\end{bmatrix}^{\mathrm{T}}$ as follows

$$\dot{X} = A_1 X u_1 + A_2 X(1-u_1)u_2 + A_3 X(1-u_1)(1-u_2)$$
$$\dot{\lambda} = -QX - A_1^{\mathrm{T}}\lambda u_1 - A_2^{\mathrm{T}}\lambda(1-u_1)u_2 - A_3^{\mathrm{T}}\lambda(1-u_1)(1-u_2).$$

The two-point boundary conditions are

$$X(0) = \begin{bmatrix}1 & 6\end{bmatrix}^{\mathrm{T}}, \qquad \lambda(2.5) = \begin{bmatrix}0 & 0\end{bmatrix}^{\mathrm{T}}$$

and the optimal switching condition is

$$\min_{k\in\{1,2,3\}} [\lambda(t)^{\mathrm{T}} A_k\, X(t)], \quad \forall t \in [0, 2.5].$$

For this problem, we chose 1001 mesh points with a mesh size of 0.0025 s. In converging to the optimal solution, *err* reduced from an initial value of 5×10^{10} to *conv*, which was chosen as 2×10^{-4}, in 19 794 iterations. The convergence related issues were more severe for this problem and can be attributed to more number of modes. This necessitated the use of a lower value for *slowc* compared to the previous simulation.

The state and co-state trajectories are plotted in Fig. 4. The phase portrait is plotted in Fig. 5 and the optimal switching schedule is shown in Fig. 6. Note that the system stays in mode S_2 for $t < t_1$, in mode S_1 for $t_1 \le t < t_2$, and switches frequently between modes S_1 and S_3 beyond t_2. The switching behavior of the system can be explained using the eigenvalues of A_1, A_2, and A_3, given below

$$A_1 = \begin{cases}\mu_{11} = -1.89\\ \mu_{12} = -11.11\end{cases}, \qquad A_2 = \begin{cases}\mu_{21} = 7.31\\ \mu_{22} = -15.31\end{cases},$$

$$A_3 = \begin{cases}\mu_{31} = -15.37\\ \mu_{32} = 6.37.\end{cases}$$

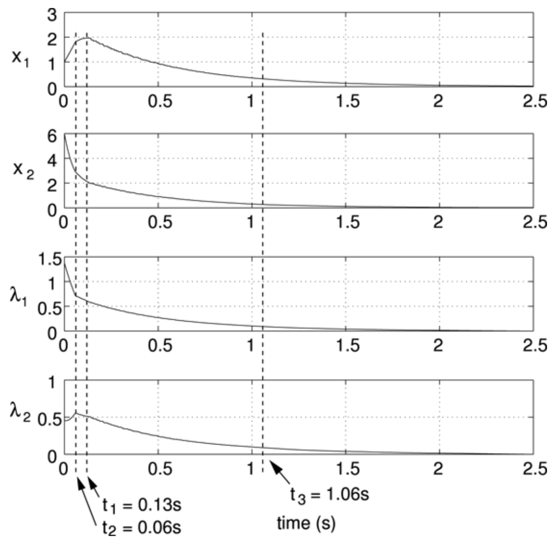

Fig. 4. State and co-state trajectories: Three-mode system.

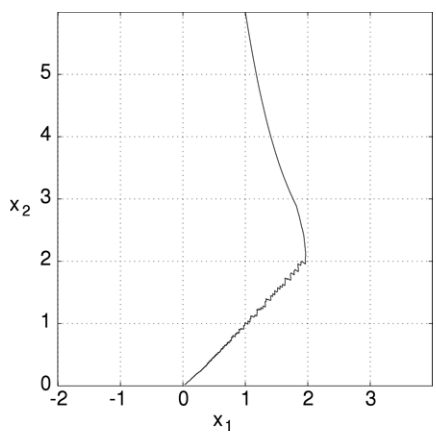

Fig. 5. Phase portrait: Three-mode system.

Beyond $t = t_3$, the system remains primarily in mode S_1, the only stable mode. This is because $x_1, x_2 \approx 0$ for $t \geq t_3 (= 1.06)$ s, as evident from Fig. 4, and there is not much advantage is switching to modes S_2 or S_3. The optimal value of J was computed to be 1.9029.

5. Conclusions

We addressed the optimal switching problem for switched linear systems in the framework of optimal control. The switching sequence, number of switchings, and switching instants were all treated as decision variables. By embedding the switched system in a larger family of systems, we were able to apply *Pontryagin's Minimum Principle* and derive the condition for optimal switching. The ensuing two-point boundary value problem was solved using the iterative relaxation method and simulation results were provided for a two-mode and a three-mode system.

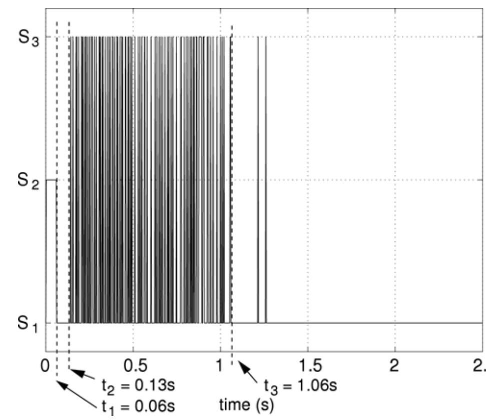

Fig. 6. Optimal switching: Three-mode system.

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References

Bemporad, A., Giua, A., & Seatzu, C. (2002a). A master-slave algorithm for the optimal control of continuous-time switched affine systems. In *Proc. IEEE conference on decision and control*: *Vol. 2*.

Bemporad, A., Giua, A., & Seatzu, C. (2002b). Synthesis of state-feedback optimal controllers for continuous-time switched linear systems. In *Proc. IEEE conference on decision and control*: *Vol. 3*.

Bengea, S. C., & DeCarlo, R. A. (2005). Optimal control of switched systems. *Automatica*, *41*, 11–27.

Branicky, M. S., Borkar, V. S., & Mitter, S. K. (1998). A unified framework for hybrid control: Model and optimal control theory. *IEEE Transactions on Automatic Control*, *43*(1), 31–45.

Cassandras, C. G., Pepyne, D. L., & Wardi, Y. (2001). Optimal control of a class of hybrid systems. *IEEE Transactions on Automatic Control*, *46*(3), 398–415.

Egerstedt, M., Wardi, Y., & Delmotte, F. (2003). Optimal control of switching times in switched dynamical systems. In *Proc. IEEE conference on decision and control*: *Vol. 3* (pp. 2138–2143).

Giua, A., Seatzu, C., & Van Der Me, C. (2001). Optimal conrol of switched autonomous linear systems. In *Proc. IEEE conference on decision and control* (pp. 2472–2477).

Press, W. H., Teukolsky, S. A., Vetterling, W. T., & Flanney, B. P. (1992). *Numerical recipes in C*. Cambridge University Press.

Sussmann, H. (1999). A maximum principle for hybrid optimal control problems. In *Proc. IEEE conference on decision and control* (pp. 425–430).

Xu, X., & Antsakalis, P. J. (2004). Optimal control of switched systems based on parameterization of the switching instants. *IEEE Transactions on Automatic Control*, *49*(1), 2–16.