Solutions of Diff Eq

Friday, August 15, 2014 10:42 PM

$$a\ddot{x} + b\dot{x} + cx = 0$$

Let $x = x_0 e^{zt}$

$$ax_0 z^2 e^{zt} + bx_0 z e^{zt} + cx_0 e^{zt} = 0$$

$$\Rightarrow x_0 e^{zt}\left[az^2 + bz + c\right] = 0$$

$\therefore x = x_0 e^{zt}$ is indeed a solution if z satisfies

$$az^2 + bz + c = 0$$

↳ characteristic equation

[Note z looks similar to the Laplace var. s]

Example: $\ddot{x} + 3\dot{x} + 2x = 0$

Let the solution be $x = x_0 e^{zt}$

$\therefore z$ must satisfy $z^2 + 3z + 2 = 0$

$\Rightarrow z_{1,2} = -1$ & -2.

the two solutions are :-

$$x_1 = x_{01} e^{-t} \quad \& \quad x_2 = x_{02} e^{-2t}$$

$\therefore$ Complete solution is :—

$$x(t) = x_{01} e^{-t} + x_{02} e^{-2t}$$

Let the given conditions be

$$x(0) = 1 \quad \& \quad \dot{x}(0) = 0$$

$\therefore 1 = x_{01} + x_{02}$ & $0 = -x_{01} - 2x_{02}$

$\Rightarrow x_{01} + 2x_{02} = 0 \Rightarrow 1 + x_{02} = 0$

$\Rightarrow x_{02} = -1$ & $x_{01} = 2$

$\therefore$ numeric solution is :-

$x(t) = 2e^{-t} - e^{-2t}$

Now, what happens for repeated roots ?

Let's say that the diff. eq. is

$\ddot{x} + 2\dot{x} + x = 0$

then, we get $(\ddot{x} + \dot{x}) + (\dot{x} + x) = 0$

$\Rightarrow \dot{z} + z = 0$ where $z = x + \dot{x}$

the solution of $\dot{z} + z = 0$ is

$z = z_0 e^{-t}$ $\therefore \dot{x} + x = z_0 e^{-t}$

It is clear that the solution must have the e^{-t} term. But a solution of the

the e^{-t} term. But a solution of the form ae^{-t} will not work (can be verified)

Let the solution be $x(t) = f(t)e^{-t}$

$\therefore \dot{x} + x = f'(t)e^{-t} - f(t)e^{-t} + f(t)e^{-t}$

$\Rightarrow \dot{x} + x = f'(t)e^{-t} = z_0 e^{-t}$

$\Rightarrow f'(t) = z_0 \Rightarrow f(t) = z_0 t + c$

$\therefore x(t) = (z_0 t + c)e^{-t}$

$= [(x_0 + \dot{x}_0)t + x_0]e^{-t}$

Example :- $\ddot{x} + 4\dot{x} + 13x = 0$

Let $x = x_0 e^{zt}$

$\dot{x} = x_0 z e^{zt}$

& $\ddot{x} = x_0 z^2 e^{zt}$

$\therefore$ Diff. equation becomes :-

$$\left(z^2+4z+13\right)x_0e^{zt}=0$$

$$\therefore z^2+4z+13=0 \Rightarrow z_{1,2}=-2\pm3j$$

$\therefore$ Two solutions exist. They are :-

$$c_1e^{(-2+3j)t} \quad \& \quad c_2e^{(-2-3j)t}$$

$$\therefore x(t)=c_1e^{(-2+3j)t}+c_2e^{(-2-3j)t}$$

$$=e^{-2t}\left(c_1e^{3jt}+c_2e^{-3jt}\right)$$

Let the initial conditions be $x(0)$ & $\dot{x}(0)$. Then

$$x(0)=c_1+c_2$$

$$\& \ \dot{x}(0)=-2(c_1+c_2)+3j(c_1-c_2)$$

Note here that c_1 & c_2 need not be real. They can be complex nos.

$$\therefore c_1+c_2=x(0)$$

$$\& \quad c_1 - c_2 = \frac{\dot{x}(0) + 2x(0)}{3j}$$

$$\therefore \quad c_1 = \frac{1}{2}\left(x(0) + \frac{\dot{x}(0) + 2x(0)}{3j}\right)$$

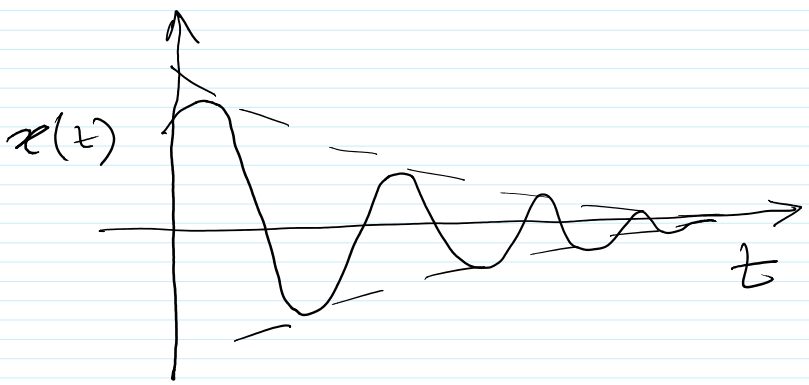
$$\& \quad c_2 = \frac{1}{2}\left(x(0) - \frac{\dot{x}(0) + 2x(0)}{3j}\right)$$

Now, recall that the solution was :-

$$x(t) = e^{-2t}\left(c_1 e^{3jt} + c_2 e^{-3jt}\right)$$

$$= e^{-2t}\left[\frac{x(0)}{2}\left(e^{3jt} + e^{-3jt}\right)\right.$$

$$\left. + \frac{\dot{x}(0) + 2x(0)}{6j}\left(e^{3jt} - e^{-3jt}\right)\right]$$

$$= e^{-2t}\left[\frac{x(0)}{2} \cdot 2\cos 3t + \frac{\dot{x}(0) + 2x(0)}{3}\sin 3t\right]$$

$$= e^{-2t}\left[x_0 \cos 3t + \frac{\dot{x}(0) + 2x(0)}{3}\sin 3t\right]$$