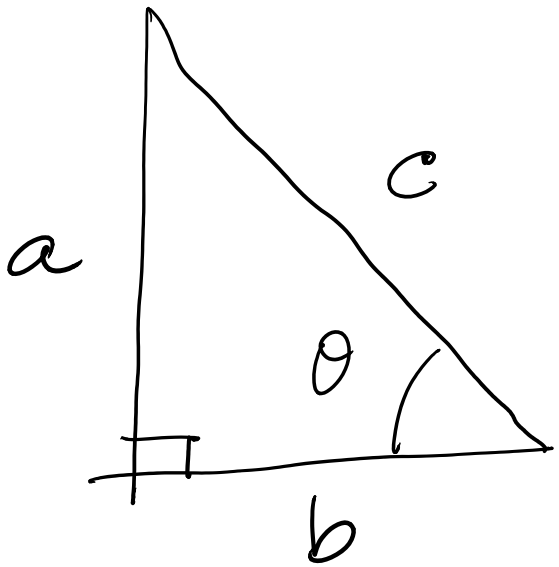
Basic results

Sunday, August 17, 2014 7:57 PM

Trig identities and the Pythagoras theorem

Consider the right triangle



$$c^2 = a^2 + b^2$$

$$\therefore\ b = c\cos\theta \ \&\ a = c\sin\theta$$

$$\therefore\ a^2 + b^2 = c^2(\sin^2\theta + \cos^2\theta)$$

But from Pyth. theorem

$$a^2 + b^2 = c^2$$

$$\therefore$$ we obtain the identity

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta + \cos^2\theta = 1$$

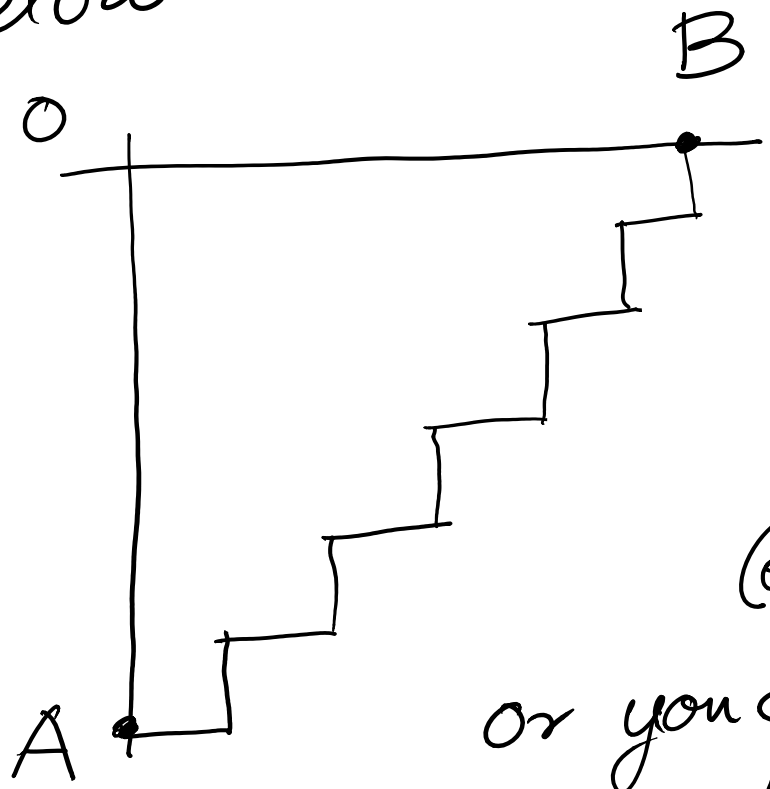
* So the above identity is basically the same as the Pyth. theorem but normalized to a right triangle with unit hypotenuse.

Since $c^2 = a^2 + b^2$

$\therefore c^2 = (a+b)^2 - 2ab$

$\therefore c < (a+b)$ where c, a, b represents the sides of a right triangle.

Question :- Say you want to go from A to B, as indicated below



Say you can go either A → O & O → B (one right turn), or you can take "inside" roads that form a diagon "alley" from A to B. Which of the two routes will be lesser distance from A to B ? Why ?

Back to trigonometry :-

What is $\sin(A+B)$ in terms of sines and cosines of A & B?

Solution 1 :- You can solve this using exponential functions.

We know that

$$e^{iA} \cdot e^{iB} = e^{i(A+B)}$$

$$\therefore (\cos A + i \sin A)(\cos B + i \sin B)$$

$$= e^{i(A+B)}$$

$$\Rightarrow (\cos A \cos B - \sin A \sin B)$$

$$\Rightarrow (\cos A \cos B - \sin A \sin B)$$

$$+ i(\sin A \cos B + \cos A \sin B)$$

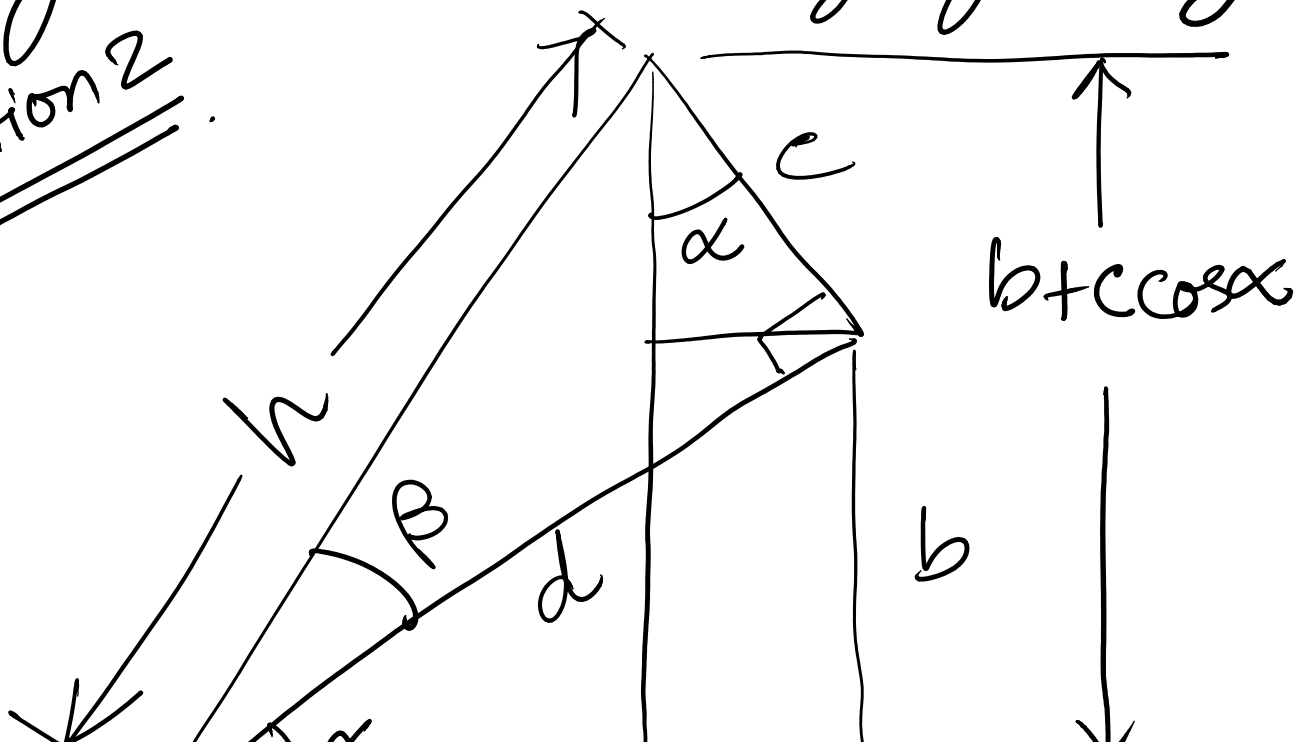
$$= \cos(A+B) + i \sin(A+B)$$

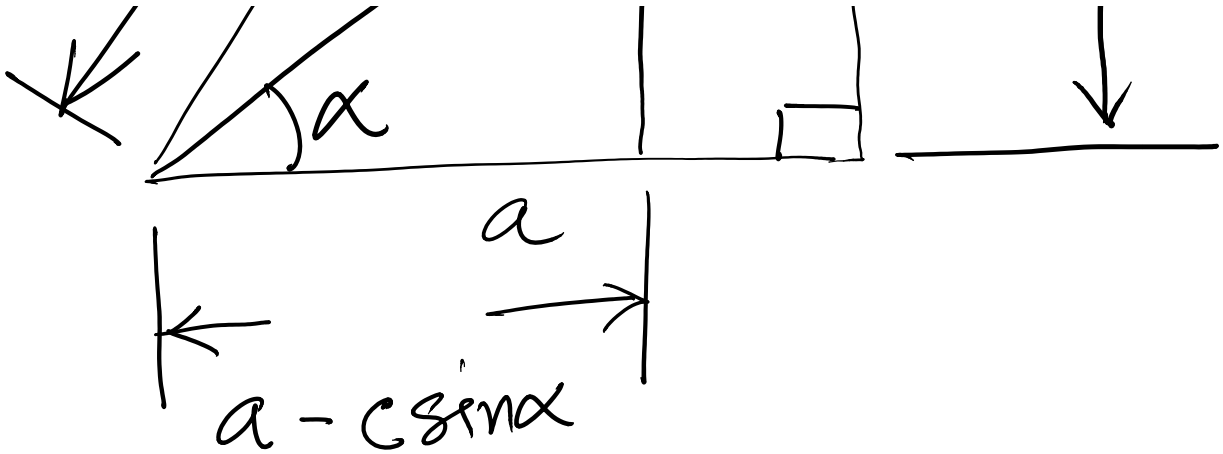
$$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\& \cos(A+B) = \cos A \cos B - \sin A \sin B$$

Can you solve this through geometry?

Solution 2





$$h^2 = a^2 + b^2 + c^2$$

$$\sin(\alpha+\beta) = \frac{b + c\cos\alpha}{h}$$

$$= \frac{b}{h} + \frac{c}{h}\cos\alpha$$

$$= \frac{b}{h} + \sin\beta\cos\alpha$$

$$= \frac{b}{d}\cdot\frac{d}{h} + \sin\beta\cos\alpha$$

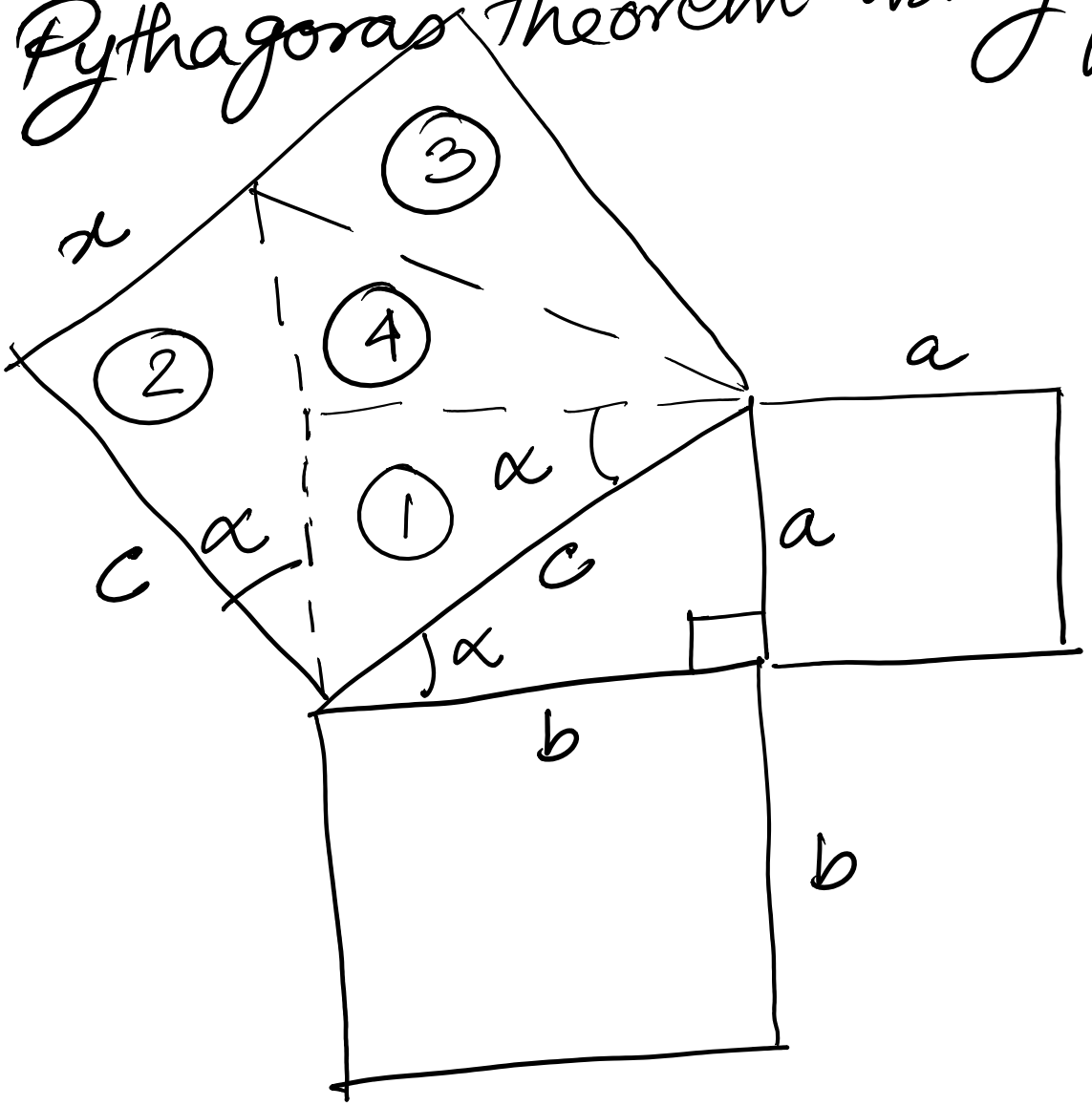
$$= \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta$$

$+ \sin\beta\cos\alpha.$

Proven through geometry.

Sample problem :- Prove the Pythagoras theorem using geometry.



Area of (1) $= \frac{1}{2}ab$

" " (2) $= \frac{1}{2}cx$

" " (3) $= \frac{1}{2}c(c-x)$

" " (4) $= \frac{1}{2}b\left(\sqrt{c^2+x^2}-a\right)$

$\therefore$ Sum $= \frac{1}{2}c^2 + \frac{1}{2}b\sqrt{c^2+x^2}$

Also, $\frac{x}{c} = \frac{a}{b} \Rightarrow x = \frac{ca}{b}$

$\therefore$ Sum $= \frac{1}{2}c^2 + \frac{1}{2}bc\sqrt{\frac{a^2+b^2}{b^2}}$

$= \frac{1}{2}c^2 + \frac{1}{2}c\sqrt{a^2+b^2}$

But sum $= c^2$

But sum = c

$$\therefore c^2 = \frac{1}{2}c^2 + \frac{1}{2}c\sqrt{a^2+b^2}$$

$$\Rightarrow \frac{c^2}{2} = \frac{c}{2}\sqrt{a^2+b^2}$$

$$\Rightarrow c = \sqrt{a^2+b^2}$$

$$\Rightarrow c^2 = a^2+b^2$$

Using the $\sin(A+B)$ & $\cos(A+B)$ identities a number of secondary identities can be derived.

Example :-

$$\sin(\theta+\pi) = \sin\theta\cos\pi + \cos\theta\sin\pi$$

$$= -\sin\theta$$

$$\sin\left(\theta+\frac{\pi}{2}\right) = \sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2}$$

$$= \cos\theta$$

& so on & so forth!

$$\sin(2\theta) = \sin(\theta+\theta)$$

$$= 2\sin\theta\cos\theta$$

$$\& \cos 2\theta = \cos(\theta+\theta)$$

$$= \cos^2\theta - \sin^2\theta$$

$$= (\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$$

this step comes from the identity

$$a^2 - b^2 = (a-b)(a+b)$$

Another identity that is often used is :—

$$a\sin\theta + b\cos\theta = \sqrt{a^2+b^2}\sin(\theta+\psi)$$

where $\cos\psi = \frac{a}{\sqrt{a^2+b^2}}$, $\sin\psi = \frac{b}{\sqrt{a^2+b^2}}$

Prove the above result