S. L. Loney, Plane trigonometry :-

If $A+B+C = \pi$, prove that

$$\cos 2A + \cos 2B - \cos 2C = 1 - 4\sin A \sin B \cos C.$$

Solution Method 1 $\quad 2C = 2\pi - 2A - 2B \quad \therefore \cos 2C = \cos(2A+2B)$

Since $\cos(2\pi - \theta) = \cos\theta$.

$\therefore$ LHS is :- $\cos 2A + \cos 2B - \cos(2A+2B)$

$= \cos 2A + \cos 2B - \cos 2A \cos 2B + \sin 2A \sin 2B$

$= \cos 2A(1-\cos 2B) + \cos 2B + 4\sin A \sin B \cos A \cos B.$

Since, $\sin 2\theta = 2\sin\theta . \cos\theta$

Note:- $\cos 2\theta = \cos^2\theta - \sin^2\theta.$

$\therefore$ LHS $= \cos 2A . 2\sin^2 B + 1 - 2\sin^2 B + 4\sin A \sin B \cos A \cos B$

$= 1 + 2\sin^2 B(\cos 2A - 1) + 4\sin A \sin B \cos A \cos B$

$= 1 - 4\sin^2 B \sin^2 A + 4\sin A \sin B \cos A \cos B$

$= 1 + 4\sin A \sin B(\cos A \cos B - \sin A \sin B)$

$= 1 + 4\sin A \sin B \cos(A+B) = 1 + 4\sin A \sin B . (-\cos C)$

Since $\cos(A+B) = \cos(\pi - C) = -\cos C.$

$\therefore$ LHS $= 1 - 4\sin A \sin B \cos C$ = RHS.

$\therefore$ proved

Method 2 Since $A+B+C=\pi$ $\therefore$ A, B, C form the angles of a triangle. Let the Δ be of sides a, b, c.

$$\therefore \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{abc}{2K}$$

where K is the area of the Δ. (See notes on Sine laws)

$$\cos 2A = 1 - 2\sin^2 A$$

$$= 1 - 2\cdot\frac{4K^2a^2}{a^2b^2c^2} = 1 - 8\frac{K^2}{b^2c^2}$$

Similarly, $\cos 2B = 1 - 8\frac{K^2}{a^2c^2}$ & $\cos 2C = 1 - 8\frac{K^2}{a^2b^2}$

$$\therefore \cos 2A + \cos 2B - \cos 2C = 1 + 8K^2\left(\frac{1}{a^2b^2} - \frac{1}{b^2c^2} - \frac{1}{c^2a^2}\right)$$

$$= 1 + 8K^2\,\frac{c^2-a^2-b^2}{a^2b^2c^2}$$

$$RHS = 1 - 4\sin A\sin B\cos C$$

$$= 1 - 4\left[\frac{2K}{bc}\cdot\frac{\cancel{2}K}{ac}\cdot\frac{a^2+b^2-c^2}{\cancel{2}ab}\right]$$

Since cosine law of Δ's says:—

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos B = \frac{a^2+c^2-b^2}{2ac}$$

$$\& \cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$\therefore R.H.S = 1 - \frac{8K^2}{a^2b^2c^2}\left(a^2+b^2-c^2\right) = 1 + \frac{8K^2}{a^2b^2c^2}\left(c^2-a^2-b^2\right)$$

$$= LHS.$$

$\therefore$ Proved

* S.L. Loney — Plane Trigonometry —

if $A+B+C = 180°$, then prove that

$$\tan\frac{A}{2}.\tan\frac{B}{2} + \tan\frac{B}{2}.\tan\frac{C}{2} + \tan\frac{C}{2}.\tan\frac{A}{2} = 1.$$

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since $C = \pi - A - B$ $\therefore$ $\frac{C}{2} = \frac{\pi}{2} - \frac{A}{2} - \frac{B}{2}$

$\therefore \tan\frac{C}{2} = \dfrac{1}{\tan\left(\frac{A}{2}+\frac{B}{2}\right)}$ Since, $\tan\left(\frac{\pi}{2}-\theta\right) = \dfrac{1}{\tan\theta}$

$\therefore$ The left. hand side (LHS) is

$$\tan\frac{A}{2}.\tan\frac{B}{2} + \frac{\tan B/2}{\tan\left(\frac{A}{2}+\frac{B}{2}\right)} + \frac{\tan A/2}{\tan\left(\frac{A}{2}+\frac{B}{2}\right)}$$

Now, Since $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta}$.

$\therefore \tan(\alpha+\beta) = \tan\alpha + \tan\beta + \tan(\alpha+\beta).\tan\alpha.\tan\beta$

$\therefore$ Going back to LHS, LHS =

$$\frac{1}{\tan\left(\frac{A}{2}+\frac{B}{2}\right)}.\left[\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\left(\frac{A}{2}+\frac{B}{2}\right).\tan\frac{A}{2}.\tan\frac{B}{2}\right]$$

$$= \frac{1}{\tan\left(\frac{A}{2}+\frac{B}{2}\right)}.\tan\left(\frac{A}{2}+\frac{B}{2}\right) = 1. = \underline{\underline{RHS.}}$$